Partial Differential Equations with Maple

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December 1997

Preface

This set of notes was written for the Tampere University of Technology's course 73131 Partial Differential Equations. It is a completely rewritten version of the second author's course notes *Osittaisdifferentiaaliyhtälöt* (TUT Course Notes No. 140,1990). The course presents the basic theory and solution techniques for the partial differential equation problems most commonly encountered in science and engineering. The student is assumed to know about linear algebra and to know something about ordinary differential equations. The textbook by Zachmanoglou and Thoe [9] is recommended as supplementary reading to these notes. Further information on the course is available at its home page

`http://matwww.ee.tut.fi/~piche/pde/index.html`

The symbolic computation program Maple (version 5.4) is used throughout these notes to solve examples and, in some cases, to carry out some steps in proofs. The source code for all the examples is available in the directory

`ftp://ftp.cc.tut.fi/pub/math/piche/pde/`

Instructions on how to configure your Web browser to open these files with Maple are given at

`http://www.maplesoft.com/technical/faq/maple/a29.html`

for instructions on how to do this.

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Chapter 1

Transformations and Canonical Forms

1.1 General Formulas for Change-of-Variables Transformations

In this section we consider two common transformations of PDEs:

- a change of independent variables, and
- a change of the dependent variable by a formula that does not involve derivatives.

We derive matrix-vector formulas for carrying out the transformations for first-order and second-order PDEs. We note that these transformations do not change the order of the PDE. The formulas we derive here will be used in later sections where we classify PDEs and find transformations that take them into their "canonical" forms.

There are special techniques for transforming coordinate independent PDE operators like grad, div, and curl from one orthogonal coordinate system to another. This is covered in vector analysis courses (e.g. [5]), and is not discussed here.

1.1.1 Change of Independent Variable

The vector of independent variables $\mathbf{x} := [x_1, \ldots, x_n]^T$ specifies a point in R^n. New independent variables $\mathbf{y} := [y_1, \ldots, y_n]^T$ are introduced by the equation set

$$\begin{aligned} x_1 &= f_1(y_1, \ldots, y_n) \\ x_2 &= f_2(y_1, \ldots, y_n) \\ &\vdots \\ x_n &= f_n(y_1, \ldots, y_n) \end{aligned}$$

which is written in vector form as

$$\mathbf{x} = \mathbf{f}(\mathbf{y})$$

The components of the jacobian matrix $\mathbf{f}_\mathbf{y}$ of the transformation are given by

$$(\mathbf{f}_\mathbf{y})_{ij} = \frac{\partial f_i}{\partial y_j}$$

Then, by the implicit function theorem, if the jacobian matrix $\mathbf{f}_\mathbf{y}$ is nonsingular at a point and $\mathbf{f}$ has continuous partial derivatives in a neighbourhood of the point, then the change of variables transformation has a continuously differentiable inverse in a neighbourhood of the point. We denote the inverse $\mathbf{g} := \mathbf{f}^{-1}$.

Applying the change of variables transformation to a scalar function $u(\mathbf{x})$ means that we are introducing a new function $v := u \circ \mathbf{f}$. The values of v are the same as the values of u, in the sense that

$$v(\mathbf{y}) = u(\mathbf{f}(\mathbf{y}))$$

We avoid the common practice of writing $u(\mathbf{y})$ to mean the composite function $u \circ \mathbf{f}$.

The chain rule gives the formula for the transformed first partial derivatives in a PDE as

$$\frac{\partial v}{\partial y_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \frac{\partial f_k}{\partial y_i}$$

or, in matrix form,

$$v_\mathbf{y} = \mathbf{f}_\mathbf{y}^T u_\mathbf{x}$$

Solving for $u_\mathbf{x}$ gives:

$$u_\mathbf{x} = \mathbf{f}_\mathbf{y}^{-T} v_\mathbf{y} \tag{1.1}$$

This is the formula for replacing the first derivative terms in a PDE by terms that use the new independent variables.

Applying the chain rule to the inverse formula

$$u(\mathbf{x}) = v(\mathbf{g}(\mathbf{x}))$$

gives

$$u_\mathbf{x} = \mathbf{g}_\mathbf{x}^T v_\mathbf{y} \tag{1.2}$$

Comparing this formula to (1.1) suggests the identity

$$\mathbf{g}_\mathbf{x} = (\mathbf{f}_\mathbf{y})^{-1} \tag{1.3}$$

which is indeed valid (Exercise 1). Thus it is not necessary to have an explicit formula for $\mathbf{g}$ in order to find its jacobian matrix; it can instead be found from $\mathbf{f}_\mathbf{y}$ by matrix inversion. This is convenient, since finding $\mathbf{g}$ can be awkward, requiring the solution of the system of possibly nonlinear equations $\mathbf{f}(\mathbf{y}) = \mathbf{x}$ for $\mathbf{y}$.

From (1.2) we see that, since the jacobian matrix $\mathbf{g}_\mathbf{x}$ is nonsingular (hence nonzero), a change of independent variables does not eliminate the first derivative terms in a first-order PDE. Thus, the order of the PDE is preserved.

Example 1
Consider the change of variables

$$x_1 = y_1, \quad x_2 = y_1/y_2$$

We define this in Maple as follows.

```
> y:=vector(2):
> f:=vector([y[1],y[1]/y[2]]);
```

$$f := \left[y_1,\ \frac{y_1}{y_2}\right]$$

(Maple displays vectors as row vectors but computes with them like column vectors.)

The jacobian matrix $\mathbf{f_y}$ is

```
> with(linalg):
> df:=jacobian(f,y);
```

$$df := \begin{bmatrix} 1 & 0 \\ \dfrac{1}{y_2} & -\dfrac{y_1}{{y_2}^2} \end{bmatrix}$$

with the matrix inverse $\mathbf{g_x} = (\mathbf{f_y})^{-1}$ given by

```
> dg:=inverse(df);
```

$$dg := \begin{bmatrix} 1 & 0 \\ \dfrac{y_2}{y_1} & -\dfrac{{y_2}^2}{y_1} \end{bmatrix}$$

The first partial derivatives $u_{\mathbf{x}}$ are then given in terms of the partial derivatives $v_{\mathbf{y}}$ by

```
> dv:=grad(v(y[1],y[2]),y);
```

$$dv := \left[\frac{\partial}{\partial y_1}\mathrm{v}(y_1, y_2), \frac{\partial}{\partial y_2}\mathrm{v}(y_1, y_2)\right]$$

```
> du:=evalm(transpose(dg) &* dv);
```

$$du := \left[\left(\frac{\partial}{\partial y_1}\mathrm{v}(y_1, y_2)\right) + \frac{y_2\left(\dfrac{\partial}{\partial y_2}\mathrm{v}(y_1, y_2)\right)}{y_1}, -\frac{{y_2}^2\left(\dfrac{\partial}{\partial y_2}\mathrm{v}(y_1, y_2)\right)}{y_1}\right]$$

For instance, applying this change of variables to the first-order PDE

$$\left(\frac{\partial}{\partial x_1} + \frac{x_2 - x_2^2}{x_1}\frac{\partial}{\partial x_2}\right) u(x_1, x_2) = 0$$

gives a first-order PDE with constant coefficients:

```
> x:=f:
> PDE:=expand(  du[1] + (x[2]-x[2]^2)/x[1]*du[2]=0  );
```

$$PDE := \frac{\partial}{\partial y_1}\mathrm{v}(y_1, y_2) + \frac{\partial}{\partial y_2}\mathrm{v}(y_1, y_2) = 0$$

Going on to the formulas for second derivatives, we have

$$\begin{aligned}
\frac{\partial^2 v}{\partial y_i \partial y_j} &= \frac{\partial}{\partial y_i}\left(\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\frac{\partial f_k}{\partial y_j}\right) \\
&= \sum_{k=1}^{n}\left(\frac{\partial}{\partial y_i}\frac{\partial u}{\partial x_k}\right)\frac{\partial f_k}{\partial y_j} + \sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\frac{\partial}{\partial y_i}\frac{\partial f_k}{\partial y_j} \\
&= \sum_{k=1}^{n}\left(\sum_{l=1}^{n}\frac{\partial}{\partial x_l}\frac{\partial u}{\partial x_k}\frac{\partial f_l}{\partial y_i}\right)\frac{\partial f_k}{\partial y_j} + \sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\frac{\partial^2 f_k}{\partial y_i \partial y_j} \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^2 u}{\partial x_k \partial x_l}\frac{\partial f_l}{\partial y_i}\frac{\partial f_k}{\partial y_j} + \sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\frac{\partial^2 f_k}{\partial y_i \partial y_j}
\end{aligned}$$

In matrix form this is

$$v_{\mathbf{yy}} = \mathbf{f}_{\mathbf{y}}^T u_{\mathbf{xx}} \mathbf{f}_{\mathbf{y}} + \sum_{k=1}^{n}\frac{\partial u}{\partial x_k}(f_k)_{\mathbf{yy}} \tag{1.4}$$

where $v_{\mathbf{yy}}$ denotes the *hessian* of v, whose i, jth element is

$$\frac{\partial^2 v}{\partial y_i \partial y_j}$$

The hessians $u_{\mathbf{xx}}$ and $(f_k)_{\mathbf{yy}}$ are defined similary.

Applying the chain rule to (1.1) gives the formula

$$u_{\mathbf{xx}} = \mathbf{g}_{\mathbf{x}}^T v_{\mathbf{yy}} \mathbf{g}_{\mathbf{x}} + \sum_{k=1}^{n}\frac{\partial v}{\partial y_k}(g_k)_{\mathbf{xx}} \tag{1.5}$$

Substituting (1.2) into (1.4) and solving for $u_{\mathbf{xx}}$ gives

$$u_{\mathbf{xx}} = \mathbf{g}_{\mathbf{x}}^T \left(v_{\mathbf{yy}} - \sum_{k=1}^{n}(\mathbf{g}_{\mathbf{x}}^T v_{\mathbf{y}})_k (f_k)_{\mathbf{yy}}\right)\mathbf{g}_{\mathbf{x}} \tag{1.6}$$

From this we can see that a change of independent variables does not change the order of a second-order PDE, since $\mathbf{g}_{\mathbf{x}}$ is not identically zero.

Introducing the notations $\mathbf{H}_k$ for $(f_k)_{\mathbf{yy}}$ and $\mathbf{e}_k$ for the kth column of the identity matrix, equation (1.6) can be rewritten as

$$u_{\mathbf{xx}} = \mathbf{g}_{\mathbf{x}}^T \left(v_{\mathbf{yy}} - \sum_{k=1}^{n}(\mathbf{e}_k^T \mathbf{g}_{\mathbf{x}}^T v_{\mathbf{y}})\mathbf{H}_k\right)\mathbf{g}_{\mathbf{x}} \tag{1.7}$$

Example 1 (continued)
Carrying on with the example change of variables described earlier, we compute the hessians $\mathbf{H}_1$ and $\mathbf{H}_2$ as follows.

```
> for k from 1 to 2 do H[k]:=hessian(f[k],y) od;
```

$$H_1 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$H_2 := \begin{bmatrix} 0 & -\frac{1}{{y_2}^2} \\ -\frac{1}{{y_2}^2} & 2\frac{y_1}{{y_2}^3} \end{bmatrix}$$

The second partial derivatives $u_{\mathbf{xx}}$ are then given in terms of the partial derivatives $v_{\mathbf{yy}}$ by

```
> ddv:=hessian(v(y[1],y[2]),y);
```

$$ddv := \begin{bmatrix} \frac{\partial^2}{\partial {y_1}^2}\mathrm{v}(y_1, y_2) & \frac{\partial^2}{\partial y_1 \partial y_2}\mathrm{v}(y_1, y_2) \\ \frac{\partial^2}{\partial y_1 \partial y_2}\mathrm{v}(y_1, y_2) & \frac{\partial^2}{\partial {y_2}^2}\mathrm{v}(y_1, y_2) \end{bmatrix}$$

```
> ddu:=evalm(transpose(dg)&*(ddv
>    -sum('du[k]*H[k]','k'=1..2))&*dg):
```

$$ddu := \left[\left(\frac{\partial^2}{\partial {y_1}^2}\mathrm{v}(y_1, y_2)\right) + 2\frac{y_2\left(\frac{\partial^2}{\partial y_1 \partial y_2}\mathrm{v}(y_1, y_2)\right)}{y_1} + \frac{{y_2}^2\left(\frac{\partial^2}{\partial {y_2}^2}\mathrm{v}(y_1, y_2)\right)}{{y_1}^2}, \right.$$
$$\left. -\frac{{y_2}^2\left(\frac{\partial^2}{\partial y_1 \partial y_2}\mathrm{v}(y_1, y_2)\right)}{y_1} - \frac{{y_2}^2\left(\frac{\partial}{\partial y_2}\mathrm{v}(y_1, y_2)\right)}{{y_1}^2} - \frac{{y_2}^3\left(\frac{\partial^2}{\partial {y_2}^2}\mathrm{v}(y_1, y_2)\right)}{{y_1}^2}\right]$$
$$\left[-\frac{{y_2}^2\left(\frac{\partial^2}{\partial y_1 \partial y_2}\mathrm{v}(y_1, y_2)\right)}{y_1} - \frac{{y_2}^2\left(\frac{\partial}{\partial y_2}\mathrm{v}(y_1, y_2)\right)}{{y_1}^2} - \frac{{y_2}^3\left(\frac{\partial^2}{\partial {y_2}^2}\mathrm{v}(y_1, y_2)\right)}{{y_1}^2}, \right.$$
$$\left. \frac{{y_2}^4\left(\frac{\partial^2}{\partial {y_2}^2}\mathrm{v}(y_1, y_2)\right)}{{y_1}^2} + 2\frac{{y_2}^3\left(\frac{\partial}{\partial y_2}\mathrm{v}(y_1, y_2)\right)}{{y_1}^2}\right]$$

For instance, applying this change of variables to the second-order PDE

$$\left\{\frac{\partial^2}{\partial x_1^2} + 2\frac{x_2}{x_1}\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{x_2^2}{x_1^2}\left((1+x_2^2)\frac{\partial^2}{\partial x_2^2} + 2x_2\frac{\partial}{\partial x_2}\right)\right\} u(x_1, x_2) = 0$$

gives a second-order PDE with constant coefficients:

```
> PDE:=expand(  ddu[1,1] + 2*x[2]/x[1]*ddu[1,2]
>     + x[2]^2/x[1]^2*((1+x[2]^2)*ddu[2,2]
>     + 2*x[2]*du[2])=0  );
```

$$PDE := \left(\frac{\partial^2}{\partial {y_1}^2}\,\mathrm{v}(y_1,\ y_2)\right) + \left(\frac{\partial^2}{\partial {y_2}^2}\,\mathrm{v}(y_1,\ y_2)\right) = 0$$

There is a Maple function, `DEtools[PDEchangecoords]`, that does all these calculations automatically. To use it, we first enter the change of variables transformation, which we call `Ex1`, into Maple.

```
> readlib(addcoords):
> addcoords(Ex1,[y[1],y[2]],[y[1],y[1]/y[2]]);
```

Warning: not an orthogonal coordinate system - no scale factors calculated.

Next, enter the PDE and apply the change of variables.

```
> x:=vector(2):
> u:='u':
> du:=grad(u(x[1],x[2]),x):
> ddu:=hessian(u(x[1],x[2]),x):
> PDE:=ddu[1,1] + 2*x[2]/x[1]*ddu[1,2] + x[2]^2/x[1]^2*
>     ((1+x[2]^2)*ddu[2,2]+2*x[2]*du[2])=0  ;
```

$$PDE := \left(\frac{\partial^2}{\partial {x_1}^2}\,\mathrm{u}(x_1,\ x_2)\right) + 2\,\frac{x_2\left(\frac{\partial^2}{\partial x_1\,\partial x_2}\,\mathrm{u}(x_1,\ x_2)\right)}{x_1} + \frac{{x_2}^2\left((1+{x_2}^2)\left(\frac{\partial^2}{\partial {x_2}^2}\,\mathrm{u}(x_1,\ x_2)\right) + 2\,x_2\left(\frac{\partial}{\partial x_2}\,\mathrm{u}(x_1,\ x_2)\right)\right)}{{x_1}^2} = 0$$

```
> expand(DEtools[PDEchangecoords](PDE,[x[1],x[2]],Ex1,
>         [y[1],y[2]]));
```

$$\left(\frac{\partial^2}{\partial {y_2}^2}\,\mathrm{u}(y_1,\ y_2)\right) + \left(\frac{\partial^2}{\partial {y_1}^2}\,\mathrm{u}(y_1,\ y_2)\right) = 0$$

This is equivalent to the result we obtained earlier, except that Maple uses the symbol u to represent the transformed variable.

1.1.2 Change of Dependent Variable

Another kind of transformation is the introduction of a new dependent variable w by an equation of the form

$$u = G(w, \mathbf{x}) \tag{1.8}$$

By the implicit function theorem, the transformation is invertible in a neighbourhood of a point provided that the partial derivative $\partial G/\partial w$ is nonzero at the point and continuous in its neighbourhood.

Applying the chain rule gives the first derivatives

$$\begin{aligned} \frac{\partial u}{\partial x_j} &= \frac{\partial G}{\partial x_j} + \frac{\partial G}{\partial w}\frac{\partial w}{\partial x_j} \\ &= G_{x_j} + G_w w_{x_j} \end{aligned}$$

The first derivative formula may be written in matrix-vector form as

$$u_{\mathbf{x}} = G_{\mathbf{x}} + G_w w_{\mathbf{x}} \tag{1.9}$$

Differentiating once more gives

$$\begin{aligned} \frac{\partial^2 u}{\partial x_i \partial x_j} &= \frac{\partial}{\partial x_i}\frac{\partial u}{\partial x_j} \\ &= \frac{\partial}{\partial x_i}\left(G_{x_j} + G_w w_{x_j}\right) \\ &= \frac{\partial G_{x_j}}{\partial x_i} + \frac{\partial G_{x_j}}{\partial w}\frac{\partial w}{\partial x_i} + \left(\frac{\partial G_w}{\partial x_i} + \frac{\partial G_w}{\partial w}\frac{\partial w}{\partial x_i}\right) w_{x_j} + G_w \frac{\partial w_{x_j}}{\partial x_i} \\ &= G_{x_i x_j} + G_{w x_j} w_{x_i} + G_{w x_i} w_{x_j} + G_{ww} w_{x_i} w_{x_j} + G_w w_{x_i x_j} \end{aligned}$$

This formula may be written in matrix-vector notation as

$$u_{\mathbf{xx}} = G_{\mathbf{xx}} + G_{0\mathbf{x}} w_{\mathbf{x}}^T + w_{\mathbf{x}} G_{0\mathbf{x}}^T + G_{00} w_{\mathbf{x}} w_{\mathbf{x}}^T + G_0 w_{\mathbf{xx}} \tag{1.10}$$

From formulas (1.9) and (1.10) it can be seen that, since $G_w \neq 0$, a change of dependent variables preserves the order of a first-order or second-order PDE.

Example 2
We consider the change of dependent variables $u = G(w, x, y, z, t)$ with

```
> G:=w*exp(b*a^2*t);
```

$$G := w\, e^{(b\, a^2\, t)}$$

The required partial derivatives of the transformation are

```
> Gw:=diff(G,w);
```

$$Gw := e^{(b\,a^2\,t)}$$

```
> with(linalg):
> X:=vector(4,[x,y,z,t]):
> dG:=grad(G,X);
```

$$dG := \left[0,\ 0,\ 0,\ w\,b\,a^2\,e^{(b\,a^2\,t)}\right]$$

```
> ddG:=hessian(G,X);
```

$$ddG := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w\,b^2\,a^4\,e^{(b\,a^2\,t)} \end{bmatrix}$$

```
> dGw:=grad(Gw,X);
```

$$dGw := \left[0,\ 0,\ 0,\ b\,a^2\,e^{(b\,a^2\,t)}\right]$$

```
> Gww:=diff(Gw,w);
```

$$Gww := 0$$

The formula for the vector of partial derivatives $u_{\mathbf{x}}$ in terms of the partial derivatives $w_{\mathbf{x}}$ is computed as:

```
> dw:=grad(w(x,y,z,t),X);
```

$$dw := \left[\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t), \frac{\partial}{\partial y}\mathrm{w}(x,y,z,t), \frac{\partial}{\partial z}\mathrm{w}(x,y,z,t), \frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right]$$

```
> du:=evalm(dG+Gw*dw);
```

$$du := \left[e^{(b\,a^2\,t)}\left(\frac{\partial}{\partial x}\,\mathrm{w}(x,y,z,t)\right),\ e^{(b\,a^2\,t)}\left(\frac{\partial}{\partial y}\,\mathrm{w}(x,y,z,t)\right),\right.$$
$$\left. e^{(b\,a^2\,t)}\left(\frac{\partial}{\partial z}\,\mathrm{w}(x,y,z,t)\right),\ w\,b\,a^2\,e^{(b\,a^2\,t)} + e^{(b\,a^2\,t)}\left(\frac{\partial}{\partial t}\,\mathrm{w}(x,y,z,t)\right)\right]$$

Similarly the formula for the matrix of partial derivatives $u_{\mathbf{xx}}$ in terms of the partial derivatives $w_{\mathbf{xx}}$ is computed (but, because of its length, is not displayed):

```
> ddw:=hessian(w(x,y,z,t),X):
> ddu:=evalm(  ddG
>      + dw&*transpose(dGw)  + dGw&*transpose(dw)
>      +Gww*(dw&*transpose(dw))  + Gw*ddw):
```

For instance, applying this change of variables to the second-order PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{a^2}\frac{\partial u}{\partial t} + bu = 0$$

and dividing through by e^{ba^2t} removes the b term:

```
> u:=G:
> PDE:=ddu[1,1]+ddu[2,2]+ddu[3,3]-du[4]/a^2+b*u=0:
> expand(PDE/exp(b*a^2*t));
```

$$\left(\frac{\partial^2}{\partial x^2}\,\mathrm{w}(x,\,y,\,z,\,t)\right)+\left(\frac{\partial^2}{\partial y^2}\,\mathrm{w}(x,\,y,\,z,\,t)\right)+\left(\frac{\partial^2}{\partial z^2}\,\mathrm{w}(x,\,y,\,z,\,t)\right)-\frac{\frac{\partial}{\partial t}\,\mathrm{w}(x,\,y,\,z,\,t)}{a^2}=0$$

The same change of variables can also be performed automatically by Maple, without going through the matrix algebra:

```
> u:='u':
> PDE:=Diff(u,x,x)+Diff(u,y,y)+Diff(u,z,z)
>       +b*u-Diff(u,t)/a^2=0;
```

$$PDE := \left(\frac{\partial^2}{\partial x^2}\,u\right)+\left(\frac{\partial^2}{\partial y^2}\,u\right)+\left(\frac{\partial^2}{\partial z^2}\,u\right)+b\,u-\frac{\frac{\partial}{\partial t}\,u}{a^2}=0$$

```
> newPDE:=subs(u=exp(b*a^2*t)*w(x,y,z,t),PDE):
> expand(value(newPDE)/exp(b*a^2*t));
```

$$\left(\frac{\partial^2}{\partial x^2}\,\mathrm{w}(x,\,y,\,z,\,t)\right)+\left(\frac{\partial^2}{\partial y^2}\,\mathrm{w}(x,\,y,\,z,\,t)\right)+\left(\frac{\partial^2}{\partial z^2}\,\mathrm{w}(x,\,y,\,z,\,t)\right)-\frac{\frac{\partial}{\partial t}\,\mathrm{w}(x,\,y,\,z,\,t)}{a^2}=0$$

1.2 Transformation of Second-Order Linear, Almost Linear and Quasilinear PDEs

In this section we show how second-order PDEs that are linear remain that way under a change of independent variables. Corresponding results are derived for PDEs that are almost linear and for PDEs that are quasilinear. We also show how almost linearity and quasilinearity are preserved by a change of dependent variables.

1.2.1 Linear PDE

A *linear* second-order PDE has the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\mathbf{x})\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x})\frac{\partial u}{\partial x_i} + c(\mathbf{x})u = d(\mathbf{x})$$

where the coefficient functions a_{ij}, b_i, c and the forcing function d are given functions of $\mathbf{x}$.

The linear second-order PDE can be written compactly as

$$\text{tr}\,(\mathbf{A}u_{\mathbf{xx}}) + \mathbf{b}^T u_{\mathbf{x}} + cu = d \tag{1.11}$$

where $\mathbf{A}$ is the matrix of second derivative coefficients a_{ij} and $\mathbf{b}$ is the vector of first derivative coefficients b_i. The notation $\text{tr}(\,\cdot\,)$ refers to the matrix trace, that is, the sum of the elements on the main diagonal. Some useful properties of trace are listed as follows.

linearity: $\text{tr}(\mathbf{A}+\mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$ and $\text{tr}(k\mathbf{A}) = k\text{tr}(\mathbf{A})$

transpose: $\text{tr}(\mathbf{A}^T) = \text{tr}(\mathbf{A})$

product: $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$

eigenvalues: $\text{tr}(\mathbf{A}) = \sum \text{eig}(\mathbf{A})$

The term $\text{tr}(\mathbf{A}u_{\mathbf{xx}})$ is called the *principal part* of the linear second-order PDE. The principal part's coefficient matrix $\mathbf{A}$ can be assumed to be symmetric without any loss of generality. This is because the principal part is unchanged if if a general coefficient matrix $\mathbf{A}$ is replaced by its symmetric part $\frac{1}{2}(\mathbf{A}+\mathbf{A}^T)$, as the following algebra shows:

$$\begin{aligned}
\text{tr}\,(\mathbf{A}u_{\mathbf{xx}}) &= \text{tr}\left(\left[\frac{1}{2}(\mathbf{A}+\mathbf{A}^T) + \frac{1}{2}(\mathbf{A}-\mathbf{A}^T)\right]u_{\mathbf{xx}}\right) \\
&= \text{tr}\left(\frac{1}{2}(\mathbf{A}+\mathbf{A}^T)u_{\mathbf{xx}}\right) + \text{tr}\left(\frac{1}{2}(\mathbf{A}-\mathbf{A}^T)u_{\mathbf{xx}}\right) \\
&= \text{tr}\left(\frac{1}{2}(\mathbf{A}+\mathbf{A}^T)u_{\mathbf{xx}}\right)
\end{aligned}$$

Here we've used the fact that the trace of the product of the skew symmetric matrix $\frac{1}{2}(\mathbf{A}-\mathbf{A}^T)$ with the symmetric matrix $u_{\mathbf{xx}}$ is zero (Problem 5).

Now we apply the formulas derived in the previous section for a change of independent variables. Applying formula (1.7) to the principal part gives

$$\begin{aligned}\text{tr}\,(\mathbf{A}u_{\mathbf{xx}}) &= \text{tr}\left(\mathbf{A}\mathbf{g}_{\mathbf{x}}^T\left[v_{\mathbf{yy}}-\sum_{k=1}^{n}(\mathbf{e}_k^T\mathbf{g}_{\mathbf{x}}^T v_{\mathbf{y}})\mathbf{H}_k\right]\mathbf{g}_{\mathbf{x}}\right)\\ &= \text{tr}\left(\mathbf{g}_{\mathbf{x}}\mathbf{A}\mathbf{g}_{\mathbf{x}}^T v_{\mathbf{yy}}\right)-\sum_{k=1}^{n}(\mathbf{e}_k^T\mathbf{g}_{\mathbf{x}}^T v_{\mathbf{y}})\text{tr}\left(\mathbf{g}_{\mathbf{x}}\mathbf{A}\mathbf{g}_{\mathbf{x}}^T\mathbf{H}_k\right)\\ &= \text{tr}\,(\mathbf{P}v_{\mathbf{yy}})-\sum_{k=1}^{n}(\mathbf{e}_k^T\mathbf{g}_{\mathbf{x}}^T v_{\mathbf{y}})\text{tr}\,(\mathbf{P}\mathbf{H}_k)\end{aligned}$$

where we've introduced $\mathbf{P}(\mathbf{y}) := \mathbf{g}_{\mathbf{x}}\mathbf{A}(\mathbf{f}(\mathbf{y}))\mathbf{g}_{\mathbf{x}}^T$. For the first derivative term we use formula (1.2) to get

$$\mathbf{b}^T u_{\mathbf{x}} = \mathbf{b}^T\mathbf{g}_{\mathbf{x}}^T v_{\mathbf{y}}$$

Putting these results together, the linear second-order PDE (1.11) is transformed to

$$\text{tr}\,(\mathbf{P}v_{\mathbf{yy}}) + \mathbf{q}^T v_{\mathbf{y}} + r\,v = s \tag{1.12}$$

where

$$\mathbf{q}(\mathbf{y}) := \mathbf{g}_{\mathbf{x}}\left(\mathbf{b}-\sum_{k=1}^{n}\text{tr}(\mathbf{P}\mathbf{H}_k)\mathbf{e}_k\right), \quad r(\mathbf{y}) := c(\mathbf{f}(\mathbf{y})), \quad s(\mathbf{y}) := d(\mathbf{f}(\mathbf{y}))$$

Since the transformed PDE (1.12) is of the same form as the original one (1.11), we see that a change of independent variables preserves the linearity of a linear second-order PDE.

Example 1 (continued)
We return to the change of variables

$$x_1 = y_1, \quad x_2 = y_1/y_2$$

applied to the linear PDE

$$\left\{\frac{\partial^2}{\partial x_1^2} + 2\frac{x_2}{x_1}\frac{\partial^2}{\partial x_1\partial x_2} + (1+x_2^2)\frac{x_2^2}{x_1^2}\frac{\partial^2}{\partial x_2^2} + 2\frac{x_2^3}{x_1^2}\frac{\partial}{\partial x_2}\right\}u(x_1,x_2) = 0$$

The coefficients of the PDE are entered as

```
> x:=vector(2):
> A:=matrix(2,2,[[1,x[2]/x[1]],
>          [x[2]/x[1],x[2]^2/x[1]^2*((1+x[2]^2))]]);
> b:=vector(2,[0,2*x[2]^3/x[1]^2]);
> c:=0: d:=0:
```

$$A := \begin{bmatrix} 1 & \frac{x_2}{x_1} \\ \frac{x_2}{x_1} & \frac{{x_2}^2\,(1+{x_2}^2)}{{x_1}^2} \end{bmatrix}$$

$$b := \left[0,\ 2\,\frac{{x_2}^3}{{x_1}^2}\right]$$

Check that these indeed give the PDE:

```
> du:=grad(u(x[1],x[2]),x);
```

$$du := \left[\frac{\partial}{\partial x_1}\,\mathrm{u}(x_1,\ x_2),\ \frac{\partial}{\partial x_2}\,\mathrm{u}(x_1,\ x_2)\right]$$

```
> ddu:=hessian(u(x[1],x[2]),x);
```

$$ddu := \begin{bmatrix} \frac{\partial^2}{\partial {x_1}^2}\,\mathrm{u}(x_1,\ x_2) & \frac{\partial^2}{\partial x_2\,\partial x_1}\,\mathrm{u}(x_1,\ x_2) \\ \frac{\partial^2}{\partial x_2\,\partial x_1}\,\mathrm{u}(x_1,\ x_2) & \frac{\partial^2}{\partial {x_2}^2}\,\mathrm{u}(x_1,\ x_2) \end{bmatrix}$$

```
> PDE:=trace(evalm(A&*ddu))+dotprod(b,du)+c*u=d;
```

$$PDE := \left(\frac{\partial^2}{\partial {x_1}^2}\,\mathrm{u}(x_1,\ x_2)\right) + 2\,\frac{x_2\left(\frac{\partial^2}{\partial x_2\,\partial x_1}\,\mathrm{u}(x_1,\ x_2)\right)}{x_1} + \frac{{x_2}^2\,(1+{x_2}^2)\left(\frac{\partial^2}{\partial {x_2}^2}\,\mathrm{u}(x_1,\ x_2)\right)}{{x_1}^2} + 2\,\frac{{x_2}^3\left(\frac{\partial}{\partial x_2}\,\mathrm{u}(x_1,\ x_2)\right)}{{x_1}^2} = 0$$

For the transformed PDE the principal part's coefficient matrix is

```
> x:=f:
> P:=map(normal,evalm(dg&*A&*transpose(dg)));
```

$$P := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the first derivative's coefficients are

```
> eye:=array(identity,1..2,1..2):
> for k from 1 to 2 do e[k]:=vector(2,i->eye[k,i])  od:
```

```
> q:=evalm((dg&*b)-sum('evalm((dg&*e[k])*
>      trace(evalm(P&*H[k])))','k'=1..2));
```

$$q := [0,\ 0]$$

The transformed PDE is given by

```
> newPDE:=trace(evalm(P&*ddv))
>         +evalm(transpose(q)&*dv)+c*v=d;
```

$$newPDE := \left(\frac{\partial^2}{\partial {y_1}^2}\,\mathrm{v}(y_1,\ y_2)\right) + \left(\frac{\partial^2}{\partial {y_2}^2}\,\mathrm{v}(y_1,\ y_2)\right) = 0$$

which is the same result as was obtained earlier.

1.2.2 Almost-Linear PDE and Quasilinear PDE

The previous results can be applied directly to two more general classes of PDEs. An *almost linear* second-order PDE has the form

$$\mathrm{tr}\,(\mathbf{A}(\mathbf{x})u_{\mathbf{xx}}) = d(\mathbf{x}, u, u_{\mathbf{x}}) \tag{1.13}$$

It is transformed by a change of independent variables to the PDE

$$\mathrm{tr}\,(\mathbf{P}(\mathbf{y})v_{\mathbf{yy}}) = d(\mathbf{f}(\mathbf{y}), v, \mathbf{g}_{\mathbf{x}}^T v_{\mathbf{y}}) + \sum_{k=1}^{n} (\mathbf{e}_k^T \mathbf{g}_{\mathbf{x}}^T v_{\mathbf{y}})\mathrm{tr}\,(\mathbf{P}(\mathbf{y})\mathbf{H}_k)$$

Thus a change of independent variables preserves the almost linearity of an almost linear second-order PDE.

A *quasilinear* second-order PDE has the form

$$\mathrm{tr}\,(\mathbf{A}(\mathbf{x}, u, u_{\mathbf{x}})u_{\mathbf{xx}}) = d(\mathbf{x}, u, u_{\mathbf{x}}) \tag{1.14}$$

It is transformed by a change of independent variables to the PDE

$$\mathrm{tr}\,(\mathbf{P}v_{\mathbf{yy}}) = d(\mathbf{f}(\mathbf{y}), v, \mathbf{g}_{\mathbf{x}}^T v_{\mathbf{y}}) + \sum_{k=1}^{n} (\mathbf{e}_k^T \mathbf{g}_{\mathbf{x}}^T v_{\mathbf{y}})\mathrm{tr}\,(\mathbf{P}\mathbf{H}_k)$$

where now

$$\mathbf{P} := \mathbf{g}_{\mathbf{x}}\mathbf{A}(\mathbf{f}(\mathbf{y}), v, \mathbf{g}_{\mathbf{x}}^T v_{\mathbf{y}})\mathbf{g}_{\mathbf{x}}^T$$

Thus a change of independent variables preserves the quasilinearity of a quasilinear second-order PDE.

A change of *dependent* variable as given by formula (1.8) doesn't preserve linearity (Exercise 6). Almost linearity and quasilinearity are preserved, however, since substituting formulas (1.8–1.10) into the PDEs (1.13) and (1.14) and dividing through by G_0 gives

$$\begin{aligned}\mathrm{tr}\,(\mathbf{A}w_{\mathbf{xx}}) \;=\; & \frac{1}{G_0} d(\mathbf{x}, G, G_{\mathbf{x}} + G_0 w_{\mathbf{x}}) \\ & - \frac{1}{G_0}\mathrm{tr}\left(\mathbf{A}(G_{\mathbf{xx}} + G_{0\mathbf{x}} w_{\mathbf{x}}^T + w_{\mathbf{x}} G_{0\mathbf{x}}^T + G_{00} w_{\mathbf{x}} w_{\mathbf{x}}^T)\right)\end{aligned}$$

1.3 Classification of Second-Order Almost Linear PDEs

We have just seen how applying a change of independent variables to a second-order almost linear PDE gives a new PDE of the same form, with the PDE's principal part's coefficient matrix $\mathbf{A}$ related to the new PDE's principal part's coefficient matrix $\mathbf{P}$ through the jacobian matrix of the change of variables $\mathbf{g_x}$ by the formula

$$\mathbf{P} = \mathbf{g_x A g_x}^T$$

This is an example of a *congruence* transformation. Two n-by-n square matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be congruent if there exists a nonsingular matrix $\mathbf{S}$ such that $\mathbf{B} = \mathbf{SAS}^T$. Congruence is an equivalence relation:

reflexivity: $\mathbf{A}$ is congruent to itself;

symmetry: if $\mathbf{A}$ is congruent to $\mathbf{B}$ then $\mathbf{B}$ is congruent to $\mathbf{A}$;

transitivity: if $\mathbf{A}$ is congruent to $\mathbf{B}$ and $\mathbf{B}$ is congruent to $\mathbf{C}$ then $\mathbf{A}$ is congruent to $\mathbf{C}$.

Congruence therefore partitions the set of coefficient matrices of PDE principal parts into equivalence classes that are invariant under a change of independent variables. The following theorem, whose proof is given in linear algebra texts, gives a criterion for recognising when two coefficient matrices are congruent.

Theorem 1.1 (Sylvester's law of inertia) *Real symmetric matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are congruent via a real congruence transformation if and only if they have the same number of positive, negative, and zero eigenvalues.*

Before applying this theorem, let's recall some related facts about eigenvalues.

- The eigenvalues of real symmetric matrices are all real.
- The number of nonzero eigenvalues of a square matrix is equal to the rank.
- A square matrix is nonsingular if and only if it has no zero eigenvalues.
- A symmetric matrix is positive (or, respectively, negative) definite if and only if all its eigenvalues are positive (resp. negative).

The following classification terminology is used for almost linear second-order PDEs.

Parabolic: **A** has one or more zero eigenvalues, that is, **A** is singular. The prototype parabolic equation is the heat equation

$$\frac{1}{\kappa}u_t = u_{xx} + u_{yy} + u_{zz}$$

with principal part's coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Elliptic: **A** has eigenvalues all positive or all negative. That is, **A** is positive definite or negative definite. The prototype elliptic equation is Laplace's equation

$$u_{xx} + u_{yy} + u_{zz} = 0$$

with principal part's coefficient matrix $\mathbf{A} = \mathbf{I}$.

Hyperbolic: **A** has one negative eigenvalue and the rest are positive, or vice versa. The prototype second-order hyperbolic equation is the wave equation

$$\frac{1}{c^2}u_{tt} = u_{xx} + u_{yy} + u_{zz}$$

with principal part's coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c^{-2} \end{bmatrix}$$

Ultrahyperbolic: **A** has no zero eigenvalues, more than one negative eigenvalue, and more than one positive eigenvalue. Ultrahyperbolic PDEs do not arise in applications.

This classification exhausts all possibilities. As a consequence of Sylvester's law of intertia, the type (parabolic, elliptic, hyperbolic, or ultrahyperbolic) of a second-order almost linear PDE at a point is invariant to a change of independent variables.

The proof of the following theorem presents an algorithm to classify a second-order quasilinear PDE without computing eigenvalues, using only elementary matrix transformations.

Theorem 1.2 *For any symmetric matrix* **A** *there exists a nonsingular matrix* **S** *such that* $\mathbf{SAS}^T$ *is diagonal with nonzero elements equal to* 1 *or* -1.

PROOF. We compute S using a symmetric version of gaussian elimination. The algorithm is as follows.

1. Start with the given symmetric matrix $\mathbf{A}$. Set $k := 1$.

2. Assuming that the rows and columns with index less than k have already been diagonalised, consider $\mathbf{A}$ to be partitioned as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & 0 & 0 \\ 0 & a_{kk} & \mathbf{A}_2^T \\ 0 & \mathbf{A}_2 & \mathbf{A}_3 \end{bmatrix}$$

   with diagonal $\mathbf{A}_1$ and symmetric $\mathbf{A}_3$.

3. If $a_{kk} = 0$ then do the following.

   (a) If the whole submatrix

$$\mathbf{A}_4 = \begin{bmatrix} a_{kk} & \mathbf{A}_2^T \\ \mathbf{A}_2 & \mathbf{A}_3 \end{bmatrix}$$

   is zero, go to step 6. Otherwise, go on to step (b).

   (b) If $\mathbf{A}_4$ has a zero diagonal but has some nonzero off-diagonal term $a_{ij} \neq 0$, then add the ith row to the jth row and add the ith column to the jth column. This operation can be represented as

$$\mathbf{A} \leftarrow \mathbf{J}_k \mathbf{A} \mathbf{J}_k^T$$

   where $\mathbf{J}_k$ is a matrix that has ones on its diagonal and a one in the jith place, and is otherwise zero. Now $\mathbf{A}_4$ has a nonzero diagonal term $a_{jj} \neq 0$. If $j = k$, this step is complete, otherwise, exchange the kth and jth rows and exchange the kth and jth columns. This operation can be represented as

$$\mathbf{A} \leftarrow \mathbf{Q}_k \mathbf{A} \mathbf{Q}_k^T$$

   where $\mathbf{Q}_k$ is a permutation matrix.

4. Now $a_{kk} \neq 0$ and we can use this as a pivot element. Define the elementary row transformation operator

$$\mathbf{E}_k = \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -a_{kk}^{-1}\mathbf{A}_2 & \mathbf{I} \end{bmatrix}$$

   and apply it to $\mathbf{A}$ in a symmetric fashion:

$$\mathbf{A} \leftarrow \mathbf{E}_k \mathbf{A} \mathbf{E}_k^T$$

   This zeros the off-diagonal terms in the kth row and column.

5. Increment k by 1. If $k < n$, go back to step 2, otherwise go on to step 6.

6. At this point **A** is a diagonal matrix, and all that remains is to normalise its nonzero elements. Define elements of the diagonal scaling matrix **D** as follows

$$d_{kk} = \begin{cases} 1/\sqrt{|a_{kk}|} & \text{if } a_{kk} \neq 0 \\ 1 & \text{if } a_{kk} = 0 \end{cases}$$

Then the operation

$$\mathbf{A} \leftarrow \mathbf{DAD}^T$$

yields a diagonal **A** whose nonzero elements are equal to 1 or -1, and the algorithm is finished.

This algorithm zeros the off-diagonal terms of **A** one row and column at a time, and ends up with a diagonal **A**. Each elementary operation of the algorithm is represented by a nonsingular matrix, and the combined effect of all the operations gives a diagonal matrix that can be represented as

$$\mathbf{E}_n\mathbf{Q}_n\mathbf{J}_n \cdots \mathbf{E}_2\mathbf{Q}_2\mathbf{J}_2\mathbf{E}_1\mathbf{Q}_1\mathbf{J}_1(\mathbf{A})\mathbf{J}_1^T\mathbf{Q}_1^T\mathbf{E}_1^T\mathbf{J}_2^T\mathbf{Q}_2^T\mathbf{E}_2^T \cdots \mathbf{J}_n^T\mathbf{Q}_n^T\mathbf{E}_n^T$$

Here the $\mathbf{E}_k$, $\mathbf{J}_k$ and $\mathbf{Q}_k$ matrices that weren't defined in the algorithm are just identity matrices. The congruence transformation that diagonalises **A** is then given by

$$\mathbf{S} = \mathbf{E}_n\mathbf{Q}_n\mathbf{J}_n \cdots \mathbf{E}_2\mathbf{Q}_2\mathbf{J}_2\mathbf{E}_1\mathbf{Q}_1\mathbf{J}_1 \quad \blacksquare$$

Example 3
Consider the following constant symmetric matrix.

```
> A:=matrix([[1,1,0,1],[1,1,2,0],[0,2,0,1],[1,0,1,0]]);
```

$$A := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Augment it with the identity matrix.

```
> with(linalg):
> Eye:=array(identity,1..4,1..4):
> AS:=augment(A,Eye);
```

$$AS := \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Use the 1,1 element as first pivot and zero the elements below it.

```
> AS:=pivot(AS,1,1,2..4);
```

$$AS := \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Apply the corresponding column operations to **A**:

```
> AS:=transpose(pivot(transpose(AS),1,1,2..4));
```

$$AS := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

The 2,2 element is not a valid pivot. Exchange rows 2 and 4 and similarly for the columns.

```
> AS:=swaprow(AS,2,4):
> AS:=swapcol(AS,2,4);
```

$$AS := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}$$

Now the 2,2 element is a valid pivot. Zero the elements of **A** below it and to the right.

```
> AS:=pivot(AS,2,2,3..4);
```

$$AS := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

```
> AS:=transpose(pivot(transpose(AS),2,2,3..4));
```

$$AS := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

Now eliminate in row and column 3.

```
> AS:=pivot(AS,3,3,4..4);
```

$$AS := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -2 \end{bmatrix}$$

```
> AS:=transpose(pivot(transpose(AS),3,3,4..4));
```

$$AS := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -2 \end{bmatrix}$$

The algorithm has converted **A** to diagonal form. **S** is the record of the effect of all the row operations. Let's extract it and verify that it does indeed diagonalise **A**.

```
> S:=submatrix(AS,1..4,5..8);
```

$$S := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -2 \end{bmatrix}$$

```
> evalm(S&*A&*transpose(S));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The algorithm in Theorem 1.2 constructs the congruence transformation that reduces the PDE's principal part's coefficient matrix **A** to a diagonal matrix whose nonzero elements are 1 or -1. An almost linear second-order PDE with such a principal part coefficient matrix is said to be in *canonical form.* Laplace's equation and the heat equation are in canonical form, and a scaling of the time variable is sufficient to put the wave equation into canonical form.

As a direct corollary of Theorem 1.2 we have

Theorem 1.3 *An almost linear PDE whose principal part's coefficient matrix* **A** *is constant can be transformed into canonical form by the constant linear change of independent variables* **y**=**Sx**.

When **A** is not constant, the transformation into canonical form given by Theorem 2 can only be applied pointwise, treating **A**(**x**) as a constant. This is useful for classification: the type of the PDE can be identified at every point. The next section discusses techniques for transforming a PDE to canonical form not just at a point, but in a neighbourhood.

1.4 Transformation to Canonical Form of Second-Order Almost Linear PDEs in Two Variables

Consider the principal part of a second-order almost linear PDE in two independent variables

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

Its eigenvalues are

$$\frac{1}{2}\left(a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4D}\right) \tag{1.15}$$

where $D := \det(\mathbf{A}) = a_{11}a_{22} - a_{12}^2$ is called the *discriminant*. From (1.15) it can be seen that the PDE can be classified on the basis of the sign of the discriminant. The PDE is

parabolic if $D = 0$ ($\mathbf{A}$ is singular),

elliptic if $D > 0$ ($\mathbf{A}$ is definite), and

hyperbolic if $D < 0$.

(It can't be ultrahyperbolic because there are only two eigenvalues.)

After a change of independent variables, the PDE principal part has coefficient matrix $\mathbf{P}$. The discriminant of the transformed PDE is

$$\det(\mathbf{P}) = \det(\mathbf{g}_\mathbf{x}\mathbf{A}\mathbf{g}_\mathbf{x}^T) = (\det \mathbf{g}_\mathbf{x})^2 \, D$$

This equation confirms that the type of a PDE is preserved by a change of independent variables.

Let's look at the coefficients of $\mathbf{P}$. In the remainder of this section we denote the original independent variables $\mathbf{x} =: [x, y]$ and the new independent variables $\mathbf{y} =: [\xi, \eta]$.

```
> A:=matrix([[a[1,1],a[1,2]],[a[1,2],a[2,2]]]):
> gx:=matrix([[Diff(g[1],x),Diff(g[1],y)],
>             [Diff(g[2],x),Diff(g[2],y)]]):
> P:=multiply(gx,A,transpose(gx)):
> p[11]:=expand(P[1,1]);
```

$$p_{11} := \left(\frac{\partial}{\partial x} g_1\right)^2 a_{1,1} + 2\left(\frac{\partial}{\partial x} g_1\right)\left(\frac{\partial}{\partial y} g_1\right) a_{1,2} + \left(\frac{\partial}{\partial y} g_1\right)^2 a_{2,2}$$

```
> p[12]:=expand(P[1,2]);
```

$$\begin{aligned} p_{12} := {} & \left(\frac{\partial}{\partial x} g_2\right)\left(\frac{\partial}{\partial x} g_1\right) a_{1,1} + \left(\frac{\partial}{\partial x} g_2\right)\left(\frac{\partial}{\partial y} g_1\right) a_{1,2} \\ & + \left(\frac{\partial}{\partial y} g_2\right)\left(\frac{\partial}{\partial x} g_1\right) a_{1,2} + \left(\frac{\partial}{\partial y} g_2\right)\left(\frac{\partial}{\partial y} g_1\right) a_{2,2} \end{aligned}$$

```
> p[22]:=expand(P[2,2]);
```

$$p_{22} := \left(\frac{\partial}{\partial x} g_2\right)^2 a_{1,1} + 2\left(\frac{\partial}{\partial x} g_2\right)\left(\frac{\partial}{\partial y} g_2\right) a_{1,2} + \left(\frac{\partial}{\partial y} g_2\right)^2 a_{2,2}$$

We want to find the change of variables functions $g_1(x, y)$ and $g_2(x, y)$ that give us **P** in canonical form. We consider the three PDE types (hyperbolic, parabolic, elliptic) separately.

1.4.1 Hyperbolic PDE

A hyperbolic PDE is said to be in *normal form* when it is of the form

$$v_{\xi\eta} = e(\xi, \eta, v, v_\xi, v_\eta) \tag{1.16}$$

The normal form's principal part has coefficient matrix

```
> A:=matrix([[0,1/2],[1/2,0]]);
```

$$A := \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

This can be put into canonical form (also called the "second normal form") via the 45° rotation given by the constant-coefficient congruence transformation

```
> S:=matrix([[1,1],[-1,1]]);
```

$$S := \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

```
> evalm(S &* A &* transpose(S) );
```

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Our strategy is to find the transformation that takes a hyperbolic PDE into its normal form, and then to apply this congruence transformation.

If both $a_{11}(x, y) = 0$ and $a_{22}(x, y) = 0$, the PDE is already in normal form. Assume therefore that $a_{11} \neq 0$; the case $a_{22} \neq 0$ follows analogously. To bring the PDE to normal form, transformation functions $g_1(x, y)$ and $g_2(x, y)$ have to be chosen in such a way that $p_{11}(x, y) = 0$ and $p_{22}(x, y) = 0$. We write these conditions in the form

$$\begin{aligned} p_{11} &= a_{11}(x, y)\left(\frac{\partial g_1}{\partial x} - m_1(x, y)\frac{\partial g_1}{\partial y}\right)\left(\frac{\partial g_1}{\partial x} - m_2(x, y)\frac{\partial g_1}{\partial y}\right) = 0 \\ p_{22} &= a_{11}(x, y)\left(\frac{\partial g_2}{\partial x} - m_1(x, y)\frac{\partial g_2}{\partial y}\right)\left(\frac{\partial g_2}{\partial x} - m_2(x, y)\frac{\partial g_2}{\partial y}\right) = 0 \end{aligned}$$

where

$$m_1 := \frac{-a_{12} + \sqrt{-D}}{a_{11}}, \quad m_2 := \frac{-a_{12} - \sqrt{-D}}{a_{11}}$$

From these conditions we see that it is sufficient to solve the two uncoupled linear first-order PDEs

$$\left.\begin{aligned} \frac{\partial g_1}{\partial x} - m_1(x,y)\frac{\partial g_1}{\partial y} &= 0 \\ \frac{\partial g_2}{\partial x} - m_2(x,y)\frac{\partial g_2}{\partial y} &= 0 \end{aligned}\right\} \tag{1.17}$$

To do this, we seek solutions of the form $g_1(x, y) = C_1$ and $g_2(x, y) = C_2$, respectively, for the ordinary differential equations (ODEs)

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= -m_1(x,y) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -m_2(x,y) \end{aligned}$$

where C_1 and C_2 are constants of integration. When C_1 is a constant, the equation $g_1(x, y) = C_1$ describes a curve in the plane, and along this curve we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}C_1 = \frac{\mathrm{d}}{\mathrm{d}x}g_1(x, y(x)) = \frac{\partial g_1}{\partial x} + \frac{\partial g_1}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\partial g_1}{\partial x} - m_1(x,y)\frac{\partial g_1}{\partial y}$$

thus the function g_1 is indeed a solution for the first part of (1.17). Similarly we can verify that g_2 is a solution for the second part of (1.17).

The jacobian of the transformation is

$$\det(\mathbf{g_x}) = \begin{vmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} \\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y} \end{vmatrix} = \frac{\partial g_1}{\partial y}\frac{\partial g_2}{\partial y}(m_1 - m_2)$$

so that the transformation is nonsingular provided that

$$\frac{\partial g_1}{\partial y} \neq 0 \ \text{ and } \ \frac{\partial g_2}{\partial y} \neq 0$$

This is also the condition for the ODE solutions $g_1(x, y) = C_1$ and $g_2(x, y) = C_2$ to be solvable for y.

The level set curves of g_1 and g_2 are called the *characteristics* of the PDE. The set of characteristics is the grid for the coordinate system in which the almost linear PDE is in normal form (1.16). In chapter 5 we'll discuss PDE solution methods that are based on characteristic curves.

Example 4
Consider the PDE $y^2 u_{xx} - x^2 u_{yy} = 0$. Its principal part coefficient matrix and discriminant are

```
> A:=matrix([[y^2,0],[0,-x^2]]);
```

$$A := \begin{bmatrix} y^2 & 0 \\ 0 & -x^2 \end{bmatrix}$$

```
> DD:=det(A);
```

$$DD := -y^2 x^2$$

The PDE is therefore hyperbolic everywhere except on the x and y axes. The characteristic equation derivatives are

```
> m[1]:=radsimp((-A[1,2]+sqrt(-DD))/A[1,1]);
```

$$m_1 := \frac{x}{y}$$

```
> m[2]:=radsimp((-A[1,2]-sqrt(-DD))/A[1,1]);
```

$$m_2 := -\frac{x}{y}$$

Solving the first characteristic equation gives

```
> dsolve(diff(y(x),x)=-m[1],y(x));
```

$$\mathrm{y}(x)^2 = -x^2 + \_C1$$

```
> solve(",_C1);
```

$$\mathrm{y}(x)^2 + x^2$$

```
> g[1]:=subs(y(x)=y,");
```

$$g_1 := y^2 + x^2$$

Solving the second characteristic equation gives

```
> dsolve(diff(y(x),x)=-m[2],y(x));
```

$$\mathrm{y}(x)^2 = x^2 + \_C1$$

```
> solve(",_C1);
```

$$\mathrm{y}(x)^2 - x^2$$

```
> g[2]:=subs(y(x)=y,");
```

$$g_2 := y^2 - x^2$$

We verify that this change of variables gives the normal form, using formula (1.5).

```
> gx:=jacobian(vector([g[1],g[2]]),[x,y]);
```

$$gx := \begin{bmatrix} 2x & 2y \\ -2x & 2y \end{bmatrix}$$

```
> dv:=grad(v(xi,eta),[xi,eta]):
> ddv:=hessian(v(xi,eta),[xi,eta]):
> ddu:=evalm(transpose(gx)&*ddv&*gx
>          +sum('dv[k]*hessian(g[k],[x,y])','k'=1..2)):
> PDE:=expand(trace(evalm(A&*ddu)))=0;
```

$$PDE := -16\,y^2 x^2 \left(\frac{\partial^2}{\partial\xi\,\partial\eta}\,\mathrm{v}(\xi,\ \eta)\right) - 2\,y^2 \left(\frac{\partial}{\partial\xi}\,\mathrm{v}(\xi,\ \eta)\right) + 2\,y^2 \left(\frac{\partial}{\partial\eta}\,\mathrm{v}(\xi,\ \eta)\right) + 2\,x^2 \left(\frac{\partial}{\partial\xi}\,\mathrm{v}(\xi,\ \eta)\right) + 2\,x^2 \left(\frac{\partial}{\partial\eta}\,\mathrm{v}(\xi,\ \eta)\right) = 0$$

Now we want to replace the x and y values by the new coordinates ξ and η. Since the new coordinates are defined with squares, the inverse formula would involve awkward radicals. However, since only squares appear in the PDE, we can work directly with them.

```
> solve({xi=g[1],eta=g[2]},{x^2,y^2});
```

$$\{x^2 = -\frac{1}{2}\,\eta + \frac{1}{2}\,\xi,\ y^2 = \frac{1}{2}\,\eta + \frac{1}{2}\,\xi\}$$

```
> NF:=collect(expand(subs(",PDE)),diff);
```

$$NF := -2 \left(\frac{\partial}{\partial\xi}\mathrm{v}(\xi,\eta)\right)\eta + 2 \left(\frac{\partial}{\partial\eta}\mathrm{v}(\xi,\eta)\right)\xi + \left(-4\,\xi^2 + 4\,\eta^2\right)\frac{\partial^2}{\partial\eta\partial\xi}\mathrm{v}(\xi,\eta) = 0$$

Dividing this through by $4(\eta^2 - \xi^2)$ gives the normal form.

```
> collect(NF/(4*eta^2-4*xi^2),diff);
```

$$-2\,\frac{\left(\frac{\partial}{\partial\xi}v(\xi,\eta)\right)\eta}{4\,\eta^2 - 4\,\xi^2} + 2\,\frac{\left(\frac{\partial}{\partial\eta}v(\xi,\eta)\right)\xi}{4\,\eta^2 - 4\,\xi^2} + \frac{\partial^2}{\partial\eta\partial\xi}v(\xi,\eta) = 0$$

Finally, we apply the change of coordinates to transform the normal form into canonical form.

```
> readlib(addcoords):
> addcoords(rot,[lambda+mu,-lambda+mu],[lambda,mu]);
> with(DEtools):
> expand(PDEchangecoords(NF,[xi,eta],rot,[lambda,mu]));
```

$$2\,\mu\left(\frac{\partial}{\partial\lambda}\,\mathrm{v}(\lambda,\,\mu)\right)-2\,\lambda\left(\frac{\partial}{\partial\mu}\,\mathrm{v}(\lambda,\,\mu)\right)-4\left(\frac{\partial^2}{\partial\mu^2}\,\mathrm{v}(\lambda,\,\mu)\right)\mu\,\lambda$$
$$+4\left(\frac{\partial^2}{\partial\lambda^2}\,\mathrm{v}(\lambda,\,\mu)\right)\mu\,\lambda=0$$

Dividing through by $4\lambda\mu$ gives the canonical form of the hyperbolic PDE.

```
> expand("/(4*lambda*mu));
```

$$\frac{1}{2}\,\frac{\frac{\partial}{\partial\lambda}\,\mathrm{v}(\lambda,\,\mu)}{\lambda}-\frac{1}{2}\,\frac{\frac{\partial}{\partial\mu}\,\mathrm{v}(\lambda,\,\mu)}{\mu}-\left(\frac{\partial^2}{\partial\mu^2}\,\mathrm{v}(\lambda,\,\mu)\right)+\left(\frac{\partial^2}{\partial\lambda^2}\,\mathrm{v}(\lambda,\,\mu)\right)=0$$

1.4.2 Parabolic PDE

In a parabolic PDE, one of the principal part's diagonal elements a_{11} or a_{22} has to be nonzero, otherwise, since $D = a_{11}a_{22} - a_{12}^2 = 0$, the principal part would be zero. In the following we assume $a_{11} \neq 0$; the case $a_{22} \neq 0$ is analogous. Our strategy is to find a coordinate transformation that makes $p_{22} = 0$; the off-diagonal terms p_{12} and p_{21} will then automatically be zero, because the PDE type is preserved.

For a parabolic PDE, the two characteristic slopes m_1 and m_2 of the hyperbolic PDE reduce to a single slope

$$m := -\frac{a_{12}}{a_{11}}$$

Seeking a solution of the form $g_2(x, y) = C_2$ for the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -m(x, y)$$

gives a change of variables function $g_2(x, y)$ that annilihates p_{22}. To complete the transformation to canonical form it suffices to choose any smooth function $g_1(x, y)$ that gives a nonsingular jacobian matrix $\mathbf{g_x}$.

Example 5
Consider the PDE $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$. Its principal part coefficient matrix and discriminant are

```
> A:=matrix([[x^2,x*y],[x*y,y^2]]);
```

$$A := \begin{bmatrix} x^2 & x\,y \\ x\,y & y^2 \end{bmatrix}$$

```
> DD:=det(A);
```

$$DD := 0$$

Thus it is parabolic everywhere. The characteristic slope is

```
> m:=-A[1,2]/A[1,1];
```

$$m := -\frac{y}{x}$$

Now solve the characteristic ordinary differential equation

```
> dsolve(diff(y(x),x)=-m,y(x));
> g[2]:=subs(y(x)=y,solve(",_C1));
```

$$y(x) = x\_C1$$

$$g_2 := \frac{y}{x}$$

To complete the change of variables, set

```
> g[1]:=x;
```

$$g_1 := x$$

Finally, verify that this change of variables gives the normal form (exactly as in Example 4):

```
> X:=[x,y]:
> gx:=jacobian(vector([g[1],g[2]]),X);
```

$$gx := \begin{bmatrix} 1 & 0 \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$$

```
> Y:=[xi,eta]:
> dv:=grad(v(xi,eta),Y):
> ddv:=hessian(v(xi,eta),Y):
> ddu:=evalm(transpose(gx)&*ddv&*gx
>          +sum('dv[k]*hessian(g[k],X)','k'=1..2)):
> PDE:=expand(trace(evalm(A&*ddu)))=0;
```

$$PDE := x^2 \left(\frac{\partial^2}{\partial \xi^2} v(\xi, \eta) \right) = 0$$

Dividing through by x^2 gives the canonical form of the parabolic PDE.

1.4.3 Elliptic PDE

For an elliptic PDE, the discriminant D is positive, and the characteristic slopes m_1 and m_2 will be the complex conjugate pair

$$m_{1,2} = \frac{-a_{12} \pm i\sqrt{D}}{a_{11}}$$

The elliptic PDE thus has no real characteristic curves. We therefore seek a complex-valued function $g_1(x, y)$ such that $g_1(x, y) = C_1$ is a solution to the characteristic ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -m_1(x, y)$$

The second component of the change of variables is the complex conjugate $g_2 = \overline{g}_1$, because taking the complex conjugate of

$$0 = \frac{\partial g_1}{\partial x} - m_1 \frac{\partial g_1}{\partial y}$$

gives

$$0 = \frac{\partial \overline{g}_1}{\partial x} - \overline{m}_1 \frac{\partial \overline{g}_1}{\partial y} = \frac{\partial g_2}{\partial x} - m_2 \frac{\partial g_2}{\partial y}$$

Now the change of variables

$$\xi = g_1(x, y), \quad \eta = g_2(x, y)$$

takes the elliptic PDE into the normal form (1.16). Since ξ and η are complex conjugates, we introduce the new real variables λ and μ through the formulas

$$\lambda = \xi + \eta, \quad \mu = i(\xi - \eta)$$

This corresponds to the constant-coefficient congruence transformation

```
> A:=matrix([[0,1/2],[1/2,0]]):
> S:=matrix([[1,1],[I,-I]]):
> evalm(S&*A&*transpose(S));
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is the canonical form for elliptic PDEs.

Exercises

1. Prove identity (1.3). (*Hint:* Apply the change of independent variable formulas to the function $v = y_j$).

2. When $\mathbf{x}$ represents position in cartesian coordinates, the new independent variables $\mathbf{y}$ are said to form an *orthogonal curvilinear coordinate system* if the matrix $\mathbf{g_x g_x^T}$ is diagonal. Show that an equivalent condition is that the matrix $\mathbf{f_y^T f_y}$ be diagonal. Show that the elliptic cylindrical coordinates defined by the transformation

$$x_1 = y_1 y_2, \quad x_2 = (y_1^2 - c^2)^{1/2}(1 - y_2^2)^{1/2}$$

where c is a constant, is an orthogonal curvilinear coordinate system.

3. Show that applying the change of dependent variables

$$u = we^{cx/2 - c^2 a^2 t/4}$$

to the PDE

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{a^2}\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u(x, y, z, t) = 0$$

eliminates the c term.

4. Euler's PDE has the form

$$a_{11}x_1^2\frac{\partial^2 u}{\partial x_1^2} + a_{12}x_1x_2\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}x_2^2\frac{\partial^2 u}{\partial x_2^2} + b_1x_1\frac{\partial u}{\partial x_1} + b_2x_2\frac{\partial u}{\partial x_2} + cu = 0$$

where the a, b, c coefficients are constants. Show that it becomes a linear PDE with constant coefficients under the change of variables

$$y_1 = \log x_1, \quad y_2 = \log x_2$$

Solve this exercise using the formulas given in the text, then again using the Maple command `DEtools[PDEchangecoords]`.

5. Show that $\mathbf{A} = -\mathbf{A}^T$ and $\mathbf{B} = \mathbf{B}^T$ implies $\mathrm{tr}(\mathbf{AB}) = 0$.

6. Give an example to show that a change of dependent variable can transform a linear PDE into one that isn't linear.

7. Find the congruence transformation that takes the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 8 \\ 3 & 5 & 8 & 10 \\ 2 & 8 & 10 & -8 \end{bmatrix}$$

into canonical form. What is the type of the PDE with this principal part coefficient matrix?

8. Consider the second-order almost linear PDE in n independent variables of the special form

$$\sum_{k=1}^{n} a_{kk}(x_k) u_{x_k x_k} = d(\mathbf{x}, u, u_{\mathbf{x}})$$

Show that it can be transformed to canonical form by a change of independent variables in a region where the signs ($+$, $-$, or 0) of all the continuous coefficient functions a_{kk} remain the same.

9. Determine the regions of the plane where the PDE

$$x u_{xx} + 2x u_{xy} + (x - 1) u_{yy} = 0$$

is hyperbolic, and determine its normal form and canonical form there. Sketch the characteristic curves.

10. Determine the regions of the plane where Euler's PDE (Exercise 4) is hyperbolic, where it is parabolic, and where it is elliptic.

11. Transform the elliptic PDE

$$y^2 u_{xx} + x^2 u_{yy} = 0$$

to canonical form.

Chapter 2

Elliptic PDEs

2.1 Boundary Value Problem

2.1.1 General Concepts

In this chapter we consider the following boundary value problem (BVP), which is used to describe a variety of steady-state or equilibrium problems in physics:

$$\mathcal{L}u = d \text{ in } \Omega, \quad \mathcal{B}u = h \text{ on } \partial\Omega \tag{2.1}$$

where $\mathcal{L}$ is the linear second-order PDE operator

$$\mathcal{L}u := \text{tr}\,(\mathbf{A}u_{\mathbf{xx}}) + \mathbf{b}^T u_{\mathbf{x}} + cu$$

and the boundary condition operator $\mathcal{B}$ is a homogeneous first order linear differential operator. The PDE domain Ω is an open connected bounded subset of R^n with piecewise smooth boundary $\partial\Omega$. The coefficients of the PDE and of the boundary conditions are assumed continuous functions of $\mathbf{x}$. The coefficient matrix $\mathbf{A}$ is supposed positive definite everywhere in Ω, that is, the PDE is elliptic. A *solution* of the BVP is a function u continuous in $\bar{\Omega} := \Omega \cup \partial\Omega$, having continuous second derivative in Ω, and satisfying (2.1).

The *homogeneous BVP* associated with (2.1) is obtained by setting $d \equiv 0$ and $h \equiv 0$. From the linearity of $\mathcal{L}$ and $\mathcal{B}$ we can immediately deduce the following facts.

- The zero function is a solution (called the *trivial solution*) to the homogeneous problem;
- Any linear combination of solutions of the homogeneous problem is a solution of the homogeneous problem;
- If u is any solution of the homogeneous problem and v is a particular solution of the inhomogeneous problem then $v + \alpha u$ is a solution of the inhomogeneous problem for any constant α;

- If u and v are solutions of the nonhomogeneous problem then $u - v$ is a solution of the homogeneous problem. Thus, if the homogeneous problem has no nontrivial solutions then any solution of the inhomogeneous problem is unique.

2.1.2 Green's Identities and Self Adjoint BVPs

If $b_j \equiv \sum_i \partial a_{ij}/\partial x_i$, then the terms $\operatorname{tr}(\mathbf{A}u_{\mathbf{xx}}) + \mathbf{b}^T u_{\mathbf{x}}$ in (2.1) may be written in divergence form $\nabla \cdot (\mathbf{A}u_{\mathbf{x}})$, as the following expansion shows:

$$\begin{aligned}
\nabla \cdot (\mathbf{A}u_{\mathbf{x}}) &= \sum_i \frac{\partial}{\partial x_i} \mathbf{A}u_{\mathbf{x}} \\
&= \sum_i \frac{\partial}{\partial x_i} \left(\sum_j a_{ij} \frac{\partial u}{\partial x_j} \right) \\
&= \sum_i \sum_j a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_j \left(\sum_i \frac{\partial a_{ij}}{\partial x_i} \right) \frac{\partial u}{\partial x_j} \\
&= \operatorname{tr}(\mathbf{A}u_{\mathbf{xx}}) + \mathbf{b}^T u_{\mathbf{x}}
\end{aligned}$$

The special case of the operator $\mathcal{L}$ given by

$$\mathcal{L}u = \nabla \cdot (\mathbf{A}u_{\mathbf{x}}) + cu$$

is called the *formally self adjoint* second order elliptic PDE operator.

Theorem 2.1 (Green's First Identity) *The formally self adjoint operator $\mathcal{L}$ satisfies*

$$\int_\Omega v\mathcal{L}u \, \mathrm{d}V = \int_{\partial\Omega} (v\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n} \, \mathrm{d}S + \int_\Omega (-v_{\mathbf{x}}^T \mathbf{A}u_{\mathbf{x}} + cuv) \, \mathrm{d}V$$

where $\mathrm{d}V$ *denotes the volume element of* Ω *and* $\mathbf{n}\mathrm{d}S$ *denotes the outwardly directed surface element of* $\partial\Omega$.

PROOF. Applying the formula for the divergence of the product of the scalar field v with the vector field $\mathbf{A}u_{\mathbf{x}}$ gives

$$\begin{aligned}
\nabla \cdot (v\mathbf{A}u_{\mathbf{x}}) &= (\nabla v) \cdot (\mathbf{A}u_{\mathbf{x}}) + v\nabla \cdot (\mathbf{A}u_{\mathbf{x}}) \\
&= v_{\mathbf{x}}^T \mathbf{A}u_{\mathbf{x}} + v(\mathcal{L}u - cu)
\end{aligned}$$

Then apply Gauss's divergence theorem, or, in one dimension, the integration by parts formula. ■

The following two formulas are corollaries of (2.1).

Theorem 2.2 (Energy Formula) *The formally self adjoint operator $\mathcal{L}$ satisfies*

$$\int_\Omega u\mathcal{L}u \, \mathrm{d}V = \int_{\partial\Omega} (u\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n} \, \mathrm{d}S + \int_\Omega (-u_{\mathbf{x}}^T \mathbf{A}u_{\mathbf{x}} + cu^2) \, \mathrm{d}V$$

Theorem 2.3 (Green's Second Identity) *The formally self adjoint operator $\mathcal{L}$ satisfies*

$$\int_\Omega v\mathcal{L}u - u\mathcal{L}v \, \mathrm{d}V = \int_{\partial\Omega} (v\mathbf{A}u_{\mathbf{x}} - u\mathbf{A}v_{\mathbf{x}}) \cdot \mathbf{n} \, \mathrm{d}S$$

The BVP (2.1) is said to be *self-adjoint* when $\mathcal{L}$ is formally self adjoint and the boundary condition operator $\mathcal{B}$ is such that the right hand side of Green's second identity vanishes when $\mathcal{B}u = \mathcal{B}v = 0$.

Theorem 2.4 *The following problems are self adjoint.*

Dirichlet problem: $\mathcal{L}u = d$ *in* Ω, $u = h$ *on* $\partial\Omega$*;*

Neumann problem: $\mathcal{L}u = d$ *in* Ω, $(\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n} = h$ *on* $\partial\Omega$*;*

Robin problem: $\mathcal{L}u = d$ *in* Ω, $f(\mathbf{x})u + g(\mathbf{x})(\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n} = h$ *on* $\partial\Omega$, *with* $|f(\mathbf{x})| + |g(\mathbf{x})| > 0$ *on* $\partial\Omega$.

PROOF. The Robin problem is the most general of the three, since setting $f \equiv 1$ and $g \equiv 0$ gives the Dirichlet problem, and setting $f \equiv 0$ and $g \equiv 1$ gives the Neumann problem. It thus suffices to show that the Robin problem is self adjoint. Let $\mathcal{B}u = \mathcal{B}v = 0$. At points of the boundary where $f \neq 0$, $u = -\frac{g}{f}(\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n}$, and similarly for v, so that

$$(v\mathbf{A}u_{\mathbf{x}} - u\mathbf{A}v_{\mathbf{x}}) \cdot \mathbf{n} = -\frac{g}{f}\left([(\mathbf{A}v_{\mathbf{x}}) \cdot \mathbf{n}](\mathbf{A}u_{\mathbf{x}}) - (\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n}](\mathbf{A}v_{\mathbf{x}})\right) \cdot \mathbf{n} = 0$$

At points where $g \neq 0$, $(\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n} = -\frac{f}{g}u$, and similarly for v, so that

$$v\mathbf{A}u_{\mathbf{x}} - u\mathbf{A}v_{\mathbf{x}} = -\frac{g}{f}(vu - uv) = 0$$

Thus, the integrand on the right hand side of Green's second identity vanishes at all points of the boundary. ■

Another class of self-adjoint BVPs is given in Exercise 7.

2.2 Well-Posedness

A problem is said to be *well posed* if it has a solution, the solution is unique, and the solution depends continuously on data such forcing function, boundary values, coefficients, and domain shape. In this section we give some uniqueness and continuity results using two approaches, the maximum principle and the energy formula.

2.2.1 Maximum Principle

Theorem 2.5 (Hopf's Maximum Principle) *Let $c \leq 0$ in Ω. If $\mathcal{L}u \geq 0$ in Ω then u does not have a positive local maximum in Ω. If $\mathcal{L}u \leq 0$ in Ω then u does not have a negative local minimum in Ω.*

PROOF. A proof can be found for example in [4, p.232]. Here we give the shorter proof that is possible if we make the stronger assumption $c < 0$. (A proof for the case where $\mathcal{L}$ is the laplacian operator will be given in section 2.4.) Let $\mathcal{L}u \geq 0$ in Ω, and assume u has a positive local maximum at some point $\mathbf{x}_0 \in \Omega$. Then at that point $u > 0$, $u_{\mathbf{x}} = \mathbf{0}$, and $u_{\mathbf{xx}}$ is a negative semidefinite matrix, with non-positive eigenvalues. Since $\mathbf{A}$ is positive definite at $\mathbf{x}_0$, it is congruent to the identity matrix. Let $\mathbf{S}$ be a constant nonsingular matrix such that $\mathbf{I} = \mathbf{S}\mathbf{A}(\mathbf{x}_0)\mathbf{S}^T$. Then at $\mathbf{x}_0$ we have

$$\operatorname{tr}(\mathbf{A}u_{\mathbf{xx}}) = \operatorname{tr}\left(\mathbf{S}^{-1}\mathbf{S}^{-T}u_{\mathbf{xx}}\right) = \operatorname{tr}\left(\mathbf{S}^{-T}u_{\mathbf{xx}}\mathbf{S}^{-1}\right) = \sum \operatorname{eig}\left(\mathbf{S}^{-T}u_{\mathbf{xx}}\mathbf{S}^{-1}\right) \leq 0$$

with the final inequality following from Sylvester's law of inertia (Theorem 1.1). Finally, since $c < 0$ and $u > 0$ we have $\mathcal{L}u < 0$ at $\mathbf{x}_0 \in \Omega$, which contradicts the initial premise. The proof of the second part of the theorem follows by applying the first part to $-u$. ■

The first application of the maximum principle is the following result, which says that the solution of the Dirichlet problem depends continuously on the boundary data.

Theorem 2.6 *If u is a solution of $\mathcal{L}u = d$ with $c \leq 0$ in Ω and boundary condition $u = h_1$ on $\partial\Omega$ and v solves the same PDE but with $v = h_2$ on the boundary, then* $\max_{\bar{\Omega}} |u - v| \leq \max_{\partial\Omega} |h_1 - h_2|$.

PROOF. Because of the linearity of the PDE, the difference $w := u - v$ satisfies $\mathcal{L}w = 0$ in Ω with boundary condition $w = h_1 - h_2$ on $\partial\Omega$. Let $w_{\max}$ be the maximum achieved by w on the compact set $\bar{\Omega}$. If $w_{\max} > \max|h_1 - h_2|$ then w has a positive maximum point at some point $\mathbf{x}_0 \in \Omega$ But by Theorem 2.5 this implies that $\mathcal{L}w < 0$ somewhere in Ω. From this contradiction we conclude $w_{\max} \leq \max|h_1 - h_2|$. The inequality $w_{\min} \geq -\max|h_1 - h_2|$ follows analogously. ■

Setting $h = h_1 = h_2$ in Theorem 2.6, we obtain the uniqueness theorem:

Theorem 2.7 *The solution of $\mathcal{L}u = d$ with $c \leq 0$ in Ω and boundary condition $u = h$ on $\partial\Omega$ is unique.*

Example 1
The solution of a Dirichlet problem may fail to be unique when $c > 0$. Consider the homogeneous partial differential equation and boundary conditions

```
> PDE:=diff(u(x,y),x,x)+diff(u(x,y),y,y)+2*u(x,y)=0;
> BC:=[ u(0,y)=0, u(Pi,y)=0, u(x,0)=0, u(x,Pi)=0 ]:
```

$$PDE := \left(\frac{\partial^2}{\partial x^2}\,\mathrm{u}(x,\ y)\right) + \left(\frac{\partial^2}{\partial y^2}\,\mathrm{u}(x,\ y)\right) + 2\,\mathrm{u}(x,\ y) = 0$$

The function $\sin x \sin y$ is a nontrivial solution:

```
> U:=(x,y)->sin(x)*sin(y):
> is(eval(subs(u=U,PDE)));
```

true

```
> map(is,eval(subs(u=U,BC)));
```

[*true, true, true, true*]

The following monotonicity result says that a nonnegative (respectively nonpositive) forcing function gives a nonpositive (resp. nonnegative) solution.

Theorem 2.8 *If $c \leq 0$ and $\mathcal{L}u \geq \mathcal{L}v$ in Ω with $u = v$ on $\partial\Omega$, then $u \leq v$.*

The proof is similar to that of Theorem (2.6).

Example 2
Theorem (2.8) is useful in giving upper or lower bounds on the solution. Consider the partial differential equation

$$u_{xx} + u_{yy} - u = 0$$

on $(0, 1) \times (0, 1)$ with boundary conditions

$$u(0, y) = 0,\ u(1, y) = 0,\ u(x, 0) = x(1 - x),\ u(x, 1) = 0$$

The trial solution $v = x(1 - x)(1 - y)$ satisfies the boundary conditions

```
> v:=(x,y)->x*(1-x)*(1-y):
> is(v(0,y)=0) and is(v(1,y)=0) and is(v(x,0)=x*(1-x))
>     and is(v(x,1)=0);
```

true

and it satisfies the same PDE but with a different forcing function.

```
> d:=diff(v,x,x)+diff(v,y,y)-v;
```

$$d := -2 + 2y - x(1-x)(1-y)$$

```
> factor(d);
```

$$-(x+1)(x-2)(-1+y)$$

Unfortunately Maple is not able to see that this forcing function is never larger than the original (zero) forcing function.

```
> is(d<=0);
```

FAIL

By looking at the separate factors, however, we can deduce that it is so.

```
 assume(0<=x,x<=1,0<=y,y<=1);
> is(-(x+1)*(x-2)>=0);
```

true

```
> is((y-1)<=0);
```

true

We can therefore conclude using Theorem 2.8 that $u(x, y) \leq x(1-x)(1-y)$ on the domain $(0, 1) \times (0, 1)$.

A monotonicity property also holds for boundary data:

Theorem 2.9 *If $c \leq 0$ and $\mathcal{L}u = \mathcal{L}v$ in Ω with $u \leq v$ on $\partial\Omega$, then $u \leq v$.*

This too is useful in bounding solutions (Exercise 3).

2.2.2 Uniqueness Theorems based on Energy Formula

The energy formula (Theorem 2.2) for formally self adjoint PDEs gives the following uniqueness result.

Theorem 2.10 *If $c \leq 0$, $fg \geq 0$, and f is not the zero function, then the solution of the Robin problem is unique.*

PROOF. It suffices to show that the only solution u of the associated homogeneous Robin problem is the trivial one. Since $\mathcal{L}u = 0$ the energy formula reduces to

$$\int_\Omega u_{\mathbf{x}}^T \mathbf{A} u_{\mathbf{x}} \, \mathrm{d}V = \int_{\partial\Omega} (u\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n} \, \mathrm{d}S + \int_\Omega cu^2 \, \mathrm{d}V$$

Let $\{\Gamma_1, \Gamma_2\}$ be a partition of the boundary $\partial\Omega$ with $f \neq 0$ on Γ_1 and $g \neq 0$ on Γ_2. On Γ_1 we have $u = -\frac{g}{f}(\mathbf{A}u_\mathbf{x}) \cdot \mathbf{n}$, so that the first integrand on the right hand side satisfies

$$(u\mathbf{A}u_\mathbf{x}) \cdot \mathbf{n} = -\frac{g}{f}\left[(\mathbf{A}u_\mathbf{x}) \cdot \mathbf{n}\right]^2 \leq 0$$

On Γ_2 we have $(\mathbf{A}u_\mathbf{x}) \cdot \mathbf{n} = -\frac{f}{g}u$, so there also the first integrand is nonnegative:

$$(u\mathbf{A}u_\mathbf{x}) \cdot \mathbf{n} = -\frac{f}{g}u^2 \leq 0$$

Since $c \leq 0$, the second integrand is nonpositive, and we are left with the inequality

$$\int_\Omega u_\mathbf{x}^T \mathbf{A}u_\mathbf{x}\, \mathrm{d}V \leq 0$$

Since $\mathbf{A}$ is positive definite, $u_\mathbf{x} = 0$, so u must be a constant function. Then, at a point of the boundary where $f \neq 0$, the homogeneous boundary condition reduces to $fu = 0$, so this constant must be zero. ■

Example 3
The Robin BVP may fail to be unique if $fg < 0$. The function $\sinh(x) + \cosh(x)$ is a nontrivial solution of the homogeneous PDE $u'' - u = 0$ with homogeneous boundary condition $u' - u = 0$ at $x = 0$ and $x = 1$, as the following Maple results confirm.

```
> u:=x->sinh(x)+cosh(x):
> is((D(D(u))-u)(x)=0);

                              true

> is((u-D(u))(0)=0)  and is((u-D(u))(1)=0);

                              true
```

As a special case of Theorem 2.10 we have

Theorem 2.11 *If $c \leq 0$ then the solution of the Dirichlet problem is unique.*

The Neumann problem has the following uniqueness results.

Theorem 2.12 *If $c \equiv 0$ then any solutions of the Neumann problem differ by a constant function.*

Theorem 2.13 *If $c \leq 0$ and is not the zero function, then any solution of the Neumann problem is unique.*

The proofs are similar to that of Theorem 2.10.

2.3 Green's Functions

A *singularity function* $F(\mathbf{x}, \mathbf{x}')$ corresponding to a formally self adjoint operator $\mathcal{L}$ is defined as a solution of the PDE

$$\mathcal{L}F(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \quad \mathbf{x}, \mathbf{x}' \in \Omega, \ \mathbf{x} \neq \mathbf{x}'$$

Here $\mathcal{L}$ operates with respect to $\mathbf{x}$, with $\mathbf{x}'$ treated as a constant parameter. The Dirac delta $\delta(\mathbf{x} - \mathbf{x}')$ is characterised by the property

$$\int_\Omega u(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}')\,\mathrm{d}V = u(\mathbf{x}')$$

for any smooth test function u.

An alternative characterisation of the singularity function is that $\mathcal{L}F = 0$ everywhere except at $\mathbf{x} = \mathbf{x}'$, and that

$$\begin{aligned} \lim_{\epsilon \to 0} \int_{\partial B_\epsilon} F \,\mathrm{d}S &= 0 \\ \lim_{\epsilon \to 0} \int_{\partial B_\epsilon} (\mathbf{A}F_{\mathbf{x}}) \cdot \mathbf{n} \,\mathrm{d}S &= 1 \end{aligned}$$

where B_ϵ is a ball of radius ϵ centered at $\mathbf{x}'$. For, if u is a smooth test function, the first limit implies

$$\left| \int_{\partial B_\epsilon} F(\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n} \,\mathrm{d}S \right| \leq \max_{B_\epsilon} |(\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n}| \times \left| \int_{\partial B_\epsilon} F \,\mathrm{d}S \right| \to 0$$

so that

$$\int_{\partial B_\epsilon} F(\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n} \,\mathrm{d}S \to 0$$

Similarly, the second limit implies

$$\int_{\partial B_\epsilon} u(\mathbf{A}F_{\mathbf{x}}) \cdot \mathbf{n} \,\mathrm{d}S \approx u(\mathbf{x}') \int_{\partial B_\epsilon} (\mathbf{A}F_{\mathbf{x}}) \cdot \mathbf{n} \,\mathrm{d}S \to u(\mathbf{x}')$$

If we denote $\Omega_\epsilon := \Omega - B_\epsilon$, then Green's second identity (Theorem 2.3) gives

$$\begin{aligned} \int_\Omega F\mathcal{L}u \,\mathrm{d}V &\approx \int_{\Omega_\epsilon} F\mathcal{L}u \,\mathrm{d}V \\ &= \int_{\Omega_\epsilon} (F\mathcal{L}u - u\mathcal{L}F]) \,\mathrm{d}V \\ &= \int_{\partial\Omega_\epsilon} [F(\mathbf{A}u_{\mathbf{x}}) - u(\mathbf{A}F_{\mathbf{x}})] \cdot \mathbf{n} \,\mathrm{d}V \\ &= \int_{\partial\Omega} [F(\mathbf{A}u_{\mathbf{x}}) - u(\mathbf{A}F_{\mathbf{x}})] \cdot \mathbf{n} \,\mathrm{d}S \\ &\quad - \int_{\partial B_\epsilon} [F(\mathbf{A}u_{\mathbf{x}}) - u(\mathbf{A}F_{\mathbf{x}})] \cdot \mathbf{n} \,\mathrm{d}S \\ &\approx \int_{\partial\Omega} [F(\mathbf{A}u_{\mathbf{x}}) - u(\mathbf{A}F_{\mathbf{x}})] \cdot \mathbf{n} \,\mathrm{d}S + u(\mathbf{x}') \end{aligned}$$

This is equivalent to the result obtained by a purely formal application of the Dirac delta property to Green's second identity:

$$\int_\Omega F\mathcal{L}u\,\mathrm{d}V - u(\mathbf{x}') = \int_\Omega (F\mathcal{L}u - u\mathcal{L}F)\,\mathrm{d}V = \int_{\partial\Omega} [F(\mathbf{A}u_{\mathbf{x}}) - u(\mathbf{A}F_{\mathbf{x}})]\cdot\mathbf{n}\,\mathrm{d}S$$

and this equivalence is what justifies the alternative characterisation.

A singularity function F is not unique, since if $H(\mathbf{x},\mathbf{x}')$ solves $\mathcal{L}H = 0$, then $F + H$ is also a valid singularity function. In particular, if H solves the BVP

$$\mathcal{L}H = 0 \text{ in } \Omega,\ \ \mathcal{B}H = -\mathcal{B}F \text{ on } \partial\Omega$$

then the singularity function $G := F + H$ satisfies the boundary value problem

$$\mathcal{L}G(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \text{ in } \Omega,\ \ \mathcal{B}G = 0 \text{ on } \partial\Omega \tag{2.2}$$

This particular singularity function is called the *Green's function* for the BVP (2.1).

Green's functions for self adjoint problems have the following property.

Theorem 2.14 (Reciprocity Principle) *The Green's function for a self adjoint BVP is symmetric, that is,* $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x})$.

PROOF. Let $\mathbf{y}$ and $\mathbf{y}'$ be fixed, and consider $u(\mathbf{x}) := G(\mathbf{x},\mathbf{y}')$ and $v(\mathbf{x}) := G(\mathbf{x},\mathbf{y})$. Since $\mathcal{B}u = 0$ and $\mathcal{B}v = 0$ and the problem is self adjoint, the boundary terms in Green's second identity vanish, and we have

$$\begin{aligned} 0 &= \int_\Omega (v\mathcal{L}u - u\mathcal{L}v)\,\mathrm{d}V \\ &= \int_\Omega [v(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}') - u\delta(\mathbf{x}-\mathbf{y}))\,\mathrm{d}V \\ &= v(\mathbf{y}') - u(\mathbf{y}) \\ &= G(\mathbf{y}',\mathbf{y}) - G(\mathbf{y},\mathbf{y}') \end{aligned}$$

which is the required symmetry relation. ■

Example 4
Consider the Dirichlet boundary value problem

$$u''(x) = d(x) \text{ on } (0,1), \quad u(0) = h_0,\ u(1) = h_1$$

First we find a singularity function.

```
> dsolve(diff(u(x),x,x)=Dirac(x-y),u(x));
```

$$u(x) = -\text{Heaviside}(x-y)y + \text{Heaviside}(x-y)x + \_C1 + \_C2x$$

```
> F:=unapply(subs(_C1=0,_C2=0,rhs(")),(x,y));
```

$$F := (x,\ y) \to -\text{Heaviside}(x-y)\,y + \text{Heaviside}(x-y)\,x$$

Next, find the regular part.

```
> interface(showassumed=0); assume(0<Y,Y<1);
> dsolve({diff(u(x),x,x)=0,u(0)=-F(0,Y),u(1)=-F(1,Y)}
>    ,u(x));
```

$$u(x) = (Y-1)\,x$$

```
> H:=unapply(subs(Y=y,rhs(")),(x,y));
```

$$H := (x,\ y) \to (-1+y)\,x$$

Assemble the Green's function and verify that it is symmetric, considering the cases $x > y$ and $x < y$ separately.

```
> G:=unapply(F(x,y)+H(x,y),(x,y));
```

$$G := (x,\ y) \to -\text{Heaviside}(x-y)\,y + \text{Heaviside}(x-y)\,x + (-1+y)\,x$$

```
> assume(X>Y); is(G(X,Y)=G(Y,X));
```

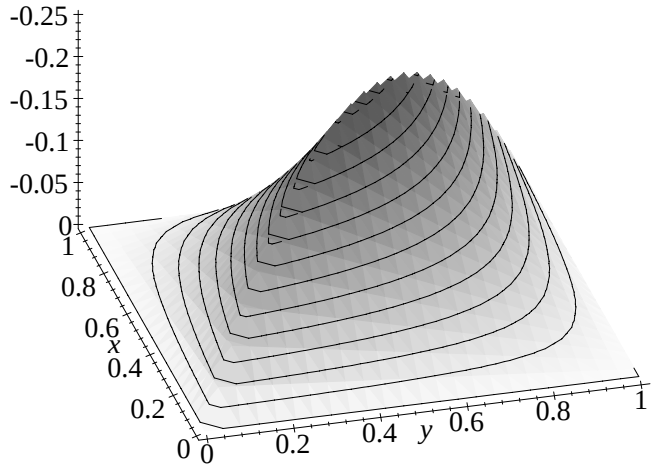
true

```
> assume(Y>X); is(G(X,Y)=G(Y,X));
```

true

Plot the Green's function and see that it is continuous and satisfies the homogeneous boundary conditions.

```
> plot3d(G(x,y),x=0..1,y=0..1,style=patchcontour,
>   axes=frame,shading=zgreyscale,orientation=[15,225]);
```



The next theorem shows how Green's function provides a solution to the boundary value problem in the form of an integral of the forcing function d and boundary function h.

Theorem 2.15 *The solution of the Robin problem with Green's function G is*

$$\begin{aligned} u(\mathbf{x}') &= \int_\Omega G(\mathbf{x},\mathbf{x}')d(\mathbf{x})\,\mathrm{d}V + \int_{\Gamma_1} \frac{h(\mathbf{x})}{f(\mathbf{x})}\left(\mathbf{A}(\mathbf{x})G_{\mathbf{x}}(\mathbf{x},\mathbf{x}')\right)\cdot\mathbf{n}\,\mathrm{d}S \\ &\quad - \int_{\Gamma_2} G(\mathbf{x},\mathbf{x}')\frac{h(\mathbf{x})}{g(\mathbf{x})}\,\mathrm{d}S \end{aligned}$$

where $\{\Gamma_1, \Gamma_2\}$ *is a partition of the boundary* $\partial\Omega$ *with* $f \neq 0$ *on* Γ_1 *and* $g \neq 0$ *on* Γ_2. *As special cases of this, the solution of the Dirichlet problem is*

$$u(\mathbf{x}') = \int_\Omega G(\mathbf{x},\mathbf{x}')d(\mathbf{x})\,\mathrm{d}V + \int_{\partial\Omega} h(\mathbf{x})\left(\mathbf{A}G_{\mathbf{x}}(\mathbf{x},\mathbf{x}')\right)\cdot\mathbf{n}\,\mathrm{d}S$$

and the solution of the Neumann problem is

$$u(\mathbf{x}') = \int_\Omega G(\mathbf{x},\mathbf{x}')d(\mathbf{x})\,\mathrm{d}V - \int_{\partial\Omega} G(\mathbf{x},\mathbf{x}')h(\mathbf{x})\,\mathrm{d}S$$

PROOF. Substituting the BVP (2.1) solution $u(\mathbf{x})$ and the Green's function into Green's second identity (Theorem 2.3) gives

$$u(\mathbf{x}') = \int_\Omega G(\mathbf{x},\mathbf{x}')d(\mathbf{x})\,\mathrm{d}V - \int_{\partial\Omega}[G(\mathbf{x},\mathbf{x}')\mathbf{A}u_{\mathbf{x}} - u\mathbf{A}G_{\mathbf{x}}(\mathbf{x},\mathbf{x}')]\cdot\mathbf{n}\,\mathrm{d}S \quad (2.3)$$

On Γ_1 we have $u = [h - g(\mathbf{A}u_{\mathbf{x}})\cdot\mathbf{n}]/f$, and the boundary integrand in (2.3) is

$$\begin{aligned} [G\mathbf{A}u_{\mathbf{x}} - u\mathbf{A}G_{\mathbf{x}}]\cdot\mathbf{n} &= \frac{1}{f}[\,fG(\mathbf{A}u_{\mathbf{x}})\cdot\mathbf{n} - \{h - g(\mathbf{A}u_{\mathbf{x}})\cdot\mathbf{n}\}(\mathbf{A}G_{\mathbf{x}})\cdot\mathbf{n}] \\ &= \frac{1}{f}[\{fG + g(\mathbf{A}G_{\mathbf{x}})\cdot\mathbf{n}\}(\mathbf{A}u_{\mathbf{x}})\cdot\mathbf{n} - h(\mathbf{A}G_{\mathbf{x}})\cdot\mathbf{n}] \\ &= -\frac{h}{f}(\mathbf{A}G_{\mathbf{x}})\cdot\mathbf{n} \end{aligned}$$

where we've used the fact that $\mathcal{B}G = fG + g(\mathbf{A}G_{\mathbf{x}})\cdot\mathbf{n} = 0$. On Γ_2 we have $(\mathbf{A}u_{\mathbf{x}})\cdot\mathbf{n} = (h - fu)/g$, and the boundary integrand in (2.3) is

$$\begin{aligned} [G\mathbf{A}u_{\mathbf{x}} - u\mathbf{A}G_{\mathbf{x}}]\cdot\mathbf{n} &= \frac{1}{g}[(h - fu)G - gu(\mathbf{A}G_{\mathbf{x}})\cdot\mathbf{n}] \\ &= \frac{1}{g}[hG - u\{fG + g(\mathbf{A}G_{\mathbf{x}})\cdot\mathbf{n}\}] \\ &= \frac{hG}{g} \end{aligned}$$

Substituting these results into (2.3) gives the required formula. ■

Example 4 (continued)
Substitute the Green's function into the solution formula for the Dirichlet

problem and verify that it satisfies the differential equation and the boundary conditions.

```
> assume(0<X,X<1,0<Y,Y<1);
> u:=int(G(X,Y)*d(X),X=0..1)+h[1]*D[1](G)(1,Y)
>     -h[0]*D[1](G)(0,Y);

u :=
    ∫_0^1 (−Heaviside(X − Y) Y + Heaviside(X − Y) X + (Y − 1) X) d(X) dX
    + h_1 Y − h_0 (Y − 1)

> is(diff(u,Y,Y)=d(Y));

                                true

is(subs(Y=0,u)=h[0]);
                                true

is(subs(Y=1,u)=h[1]);
                                true
```

If $c \equiv 0$ the Green's function for the Neumann problem (and for the Robin problem with $f \equiv 0$) is not uniquely defined. The *Neumann function* $N(\mathbf{x}, \mathbf{x}', \mathbf{x}'')$ is then used instead. It is defined formally as the solution of the problem $\mathcal{L}N(\mathbf{x}, \mathbf{x}', \mathbf{x}'') = \delta(\mathbf{x} - \mathbf{x}') - \delta(\mathbf{x} - \mathbf{x}'')$ with $\mathcal{B}N = 0$, that is, the forcing function consists of two equal and opposite impulse functions applied at the two locations $\mathbf{x}'$ and $\mathbf{x}''$. The Neumann problem solution is then

$$u(\mathbf{x}') - u(\mathbf{x}'') = \int_\Omega N(\mathbf{x}, \mathbf{x}', \mathbf{x}'') d(\mathbf{x}) \, \mathrm{d}V - \int_{\partial\Omega} N(\mathbf{x}, \mathbf{x}', \mathbf{x}'') h(\mathbf{x}) \, \mathrm{d}S$$

The solution is given in terms of the difference between the values at two locations. This is because the solution of the Neumann problem with $c \equiv 0$ is only defined to within an additive constant (Theorem 2.12).

The Green's function provides a complete solution to an elliptic boundary value problem, in much the same way that an inverse matrix provides a general solution for systems of linear equations. In later sections we present the Green's functions for a few of the most common PDE problems. Green's functions for over 500 problems are tabulated in [2].

2.4 Laplace's Equation

2.4.1 Laplacian Operator

The formally self adjoint second order elliptic operator obtained by setting $\mathbf{A} \equiv \mathbf{I}$ is called the *scalar laplacian operator*, or simply the *laplacian*, and is denoted

$$\mathcal{L}u = \Delta u := \mathrm{tr}(u_{\mathbf{xx}}) = \sum_i \frac{\partial u}{\partial x_i \partial x_j}$$

The laplacian operator is often encountered in applications. It is isotropic, i.e. does not depend on the orientation of the coordinate system (Exercise 10).

The techniques of Chapter 1 could be used to find expressions for the laplacian in different coordinate systems. However, for *orthogonal curvilinear* coordinate systems (as defined in Exercise 1.2), there are special techniques for transforming the laplacian, described in Vector Analysis textbooks such as [5]. Maple uses this technique to compute the laplacian in two and three dimensions. For example, in polar coordinates we have

```
with(linalg):
laplacian(v(r,phi),[r,phi],coords=polar);
```

$$\frac{\left(\frac{\partial}{\partial r}\,\mathrm{v}(r,\,\phi)\right)) + r\,\left(\frac{\partial^2}{\partial r^2}\,\mathrm{v}(r,\,\phi)\right) + \frac{\left(\frac{\partial^2}{\partial \phi^2}\,\mathrm{v}(r,\,\phi)\right)}{r}}{r}$$

Maple knows about 15 different orthogonal coordinate systems in two dimensions and 31 systems in three dimensions; enter `help coords` for details.

A function $u \in C^2(\Omega)$ that satisfies *Laplace's equation*

$$\Delta u = 0 \text{ in } \Omega$$

is said to be *harmonic* in Ω. The nonhomogeneous PDE associated with Laplace's equation is *Poisson's equation*

$$\Delta u = d$$

2.4.2 Poisson's Integral Formula

We start with the following key fact.

Theorem 2.16 *A symmetric singularity function for the laplacian is*

$$F(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}'|$$

in 2 dimensions and

$$F(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi} |\mathbf{x} - \mathbf{x}'|^{-1}$$

in 3 dimensions.

PROOF. For convenience we shift $\mathbf{x}'$ to the origin (see Exercise 10). The three dimensional singularity function candidate can then be written in spherical coordinates as $F = -(4\pi r)^{-1}$. Now show that F has the three properties of the alternative characterisation given on page 38. First, verify that it is harmonic:

```
> F := -1/(4*Pi*r):
> with(linalg):
> laplacian(F, [r,phi,theta], coords=spherical);
```

$$0$$

Next, show that $\lim_{\epsilon\to 0}\int_{\partial B_\epsilon} F\,\mathrm{d}S = 0$, using the fact that the surface element for the sphere is $\mathrm{d}S = \epsilon^2 \sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta$:

```
> limit(int(int(F*r^2*sin(theta),theta=0..Pi),
>      phi=0..2*Pi),r=0);
```

$$0$$

Finally, show that $\int_{\partial B_\epsilon} F_n\,\mathrm{d}S = 1$ (where $F_n := F_{\mathbf{x}}\cdot\mathbf{n}$), using the fact that $F_n = \partial F/\partial r$:

```
> int(int(diff(F,r)*r^2*sin(theta),theta=0..Pi)
>      ,phi=0..2*Pi);
```

$$1$$

For the two dimensional case the calculations are similar, except that $\mathrm{d}S = \epsilon\mathrm{d}\phi$:

```
> F := log(r)/(2*Pi):
> laplacian(F, [r,phi], coords=polar);
```

$$0$$

```
> limit(int(F*r,phi=0..2*Pi),r=0);
```

$$0$$

```
> int(diff(F,r)*r,phi=0..2*Pi);
```

$$1$$

and this completes the proof. ■

Recall that the Green's function for the Poisson's equation's Dirichlet problem is given by $G = F + H$ where H is a solution of

$$\Delta H(\mathbf{x},\mathbf{x}') = 0 \quad (\mathbf{x}\in\Omega), \qquad H(\mathbf{x},\mathbf{x}') = -F(\mathbf{x},\mathbf{x}') \quad (\mathbf{x}\in\partial\Omega) \tag{2.4}$$

Solutions of this special Dirichlet problem for various geometries can be found in the literature. For example, when $\Omega\subset\mathsf{R}^2$ is a disc with radius R

and center at the origin, the solution of (2.4) is

$$H(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \log\left(\frac{R|\mathbf{x}|}{||\mathbf{x}|^2\mathbf{x}' - R^2\mathbf{x}|}\right)$$

For a derivation see for example [9, p.204]. Here we just verify that it is indeed a solution. Substituting $|\mathbf{x}| = R$ shows that it satisfies the boundary condition. The harmonicity is verified as follows.

```
> with(linalg):
> n:=x->sqrt(dotprod(x,x)):
> x:=vector(2): y:=vector(2):
> H:=log(R*n(x)/n(evalm(dotprod(x,x)*y-R^2*x)))/2/Pi:
> is(laplacian(H,x)=0);

                              true
```

The corresponding solution for a sphere in R^3 is

$$H(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi} \frac{R|\mathbf{x}|}{||\mathbf{x}|^2\mathbf{x}' - R^2\mathbf{x}|}$$

The calculations to verify harmonicity are similar:

```
> x:=vector(3): y:=vector(3):
> H:=R*n(x)/n(evalm(dotprod(x,x)*y-R^2*x))/4/Pi:
> is(laplacian(H,x)=0);

                              true
```

Introducing polar coordinates with $r := |\mathbf{x}|$, $\rho := |\mathbf{x}'|$, and $\gamma := \angle(\mathbf{x}, \mathbf{x}')$, and using the cosine law ($|\mathbf{a} - \mathbf{b}|^2 = a^2 + b^2 - 2ab\cos\gamma$), the Green's function for the Poisson equation Dirichlet problem on the disc in R^2 can be written

$$G = \frac{1}{2\pi} \log\left(\frac{R\sqrt{r^2 + \rho^2 - 2r\rho\cos\gamma}}{\sqrt{\rho^2 r^2 + R^4 - 2R^2 r\rho\cos\gamma}}\right)$$

On the sphere in R^3 it is

$$G = \frac{1}{4\pi}\left(\frac{-1}{\sqrt{r^2 + \rho^2 - 2r\rho\cos\gamma}} + \frac{R}{\sqrt{\rho^2 r^2 + R^4 - 2R^2 r\rho\cos\gamma}}\right)$$

From Theorem 2.15, the solution of the Dirichlet problem for Laplace's equation is given by

$$u(\mathbf{x}') = \int_{\partial\Omega} h(\mathbf{x}) G_n(\mathbf{x}, \mathbf{x}')\, \mathrm{d}S \tag{2.5}$$

For the ball, $G_n(\mathbf{x}, \mathbf{x}') = \partial G/\partial r$. In two dimensions this is

```
> G:=(log(R*sqrt((r^2+rho^2-2*r*rho*cos(gamma))/
>     (rho^2*r^2+R^4-2*R^2*r*rho*cos(gamma))))/2/Pi:
> radsimp(subs(r=R,diff(G,r)));
```

$$\frac{1}{2}\,\frac{-R^2+\rho^2}{R\,(-R^2-\rho^2+2\,R\,\rho\cos(\gamma))\,\pi}$$

Using this result and the polar coordinate formulas $\cos\gamma = \cos(\phi' - \phi)$ and $\mathrm{d}S = R\,\mathrm{d}\phi$, (2.5) can be written

$$u(\rho,\phi') = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos\gamma} h(\phi)\mathrm{d}\phi$$

This is called *Poisson's integral formula*.

In three dimensions $\partial G/\partial r$ is

```
> G:=(-1/sqrt(r^2+rho^2-2*r*rho*cos(gamma))
>   +R/sqrt(rho^2*r^2+R^4-2*R^2*r*rho*cos(gamma)))/4/Pi:
> radsimp(subs(r=R,diff(G,r)));
```

$$-\frac{1}{4}\,\frac{I\,(-R^2+\rho^2)}{R\,(-R^2-\rho^2+2\,R\,\rho\cos(\gamma))^{3/2}\,\pi}$$

Substituting this result and the formula $\mathrm{d}S = R^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta$ into (2.5) gives Poisson's integral formula for the ball in three dimensions as

$$u(\rho,\theta',\phi') = \frac{R}{4\pi}\int_0^{\pi}\int_0^{2\pi} \frac{R^2-\rho^2}{(R^2+\rho^2-2R\rho\cos\gamma)^{3/2}} h(\theta,\phi)\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta$$

where $\cos\gamma = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos(\phi'-\phi)$.

2.4.3 Mean Value Property and Maximum Principle

Substituting $\rho = 0$ into Poisson's integral formula in two or three dimensions gives

Theorem 2.17 (Mean Value Property) *If u is harmonic in Ω then its value at a point $\mathbf{x}'$ is equal to its average over the surface of any ball $B \subset \Omega$ centred at $\mathbf{x}'$, that is,*

$$u(\mathbf{x}') = \frac{1}{|\partial B|}\int_{\partial B} u(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S$$

The mean value property of harmonic functions is used to prove the following alternative to Theorem 2.5.

Theorem 2.18 (Maximum Principle for the laplacian) *If u is harmonic in Ω and continuous in $\bar{\Omega}$ and if u attains its maximum or its minimum in Ω then u is constant in $\bar{\Omega}$.*

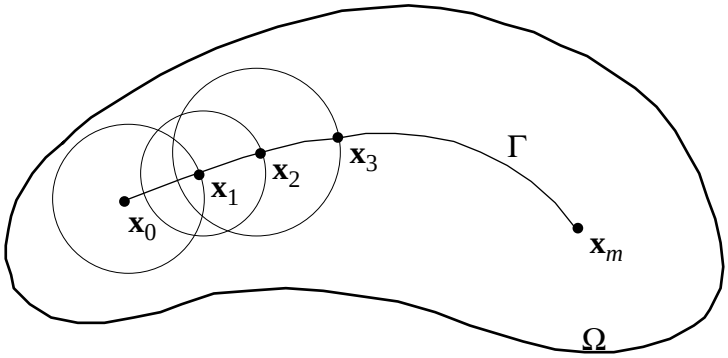


Figure 2.1: Diagram for Proof of Maximum Principle.

PROOF. Suppose u attains its maximum $M := \max_{\bar{\Omega}} u$ at a point $\mathbf{x}_0 \in \Omega$. We wish to show that at any other point $\mathbf{x}_m \in \Omega$ we must have $u(\mathbf{x}_m) = M$. Let the curve $\Gamma \subset \Omega$ connect $\mathbf{x}_0$ and $\mathbf{x}_m$, and choose the finite set of points $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_{m-1}$ on Γ to be centers of balls contained Ω, and arranged so that the point $\mathbf{x}_{i+1}$ lies on the surface ∂B_i of the ball B_i centred at the previous point $\mathbf{x}_i$. The values on ∂B_0 are all less than or equal to M. But, by the mean value property (Theorem 2.17) $u(\mathbf{x}_0)$ must be equal to the average of the values on the ball's surface, and so the surface values must all be equal to M. In particular, $u(\mathbf{x}_1) = M$. With similar arguments we obtain $u(\mathbf{x}_i) = M$ for $i = 2, 3, \ldots m$ (Figure 2.1). The proof for the minimum is similar. ■

From Theorem 2.18 we can obtain the results of section 2.2.1 on continuous dependence on boundary data and monotonicity of solutions of the Dirichlet problem.

2.4.4 Existence of Solution

This chapter has given several uniqueness results but has not yet said anything about the existence of the solution. We close the chapter with a few words about this.

The Dirichlet problem can in fact fail to have a solution if there are sharp enough "spikes" that penetrate into the domain Ω. In the absence of such spikes, however, a solution will exist; see [9, p.198] for details. Domains encountered in applications are unlikely to cause trouble in this regard.

An alternative is to replace the PDE by an integral formulation of the boundary value problem that doesn't require so much smoothness in the solution. Such variational or weak formulations are the starting point for the theory of numerical methods such as the Finite Element Method.

2.5 Eigenvalues and Eigenfunctions

2.5.1 Eigenvalues of Self-Adjoint BVP

In this section it is convenient to use some of the notation of linear vector space theory. Recall that a *scalar product* of two complex functions on Ω is

$$\langle u, v\rangle := \int_\Omega u(\mathbf{x})\bar{v}(\mathbf{x})\,\mathrm{d}V$$

where $\bar{v}$ means the complex conjugate. This scalar product has the standard scalar product properties, namely, it is

conjugate symmetric: $\langle u, v\rangle = \overline{\langle v, u\rangle}$

linear in first argument: $\langle \alpha u + \beta w, v\rangle = \alpha\langle u, v\rangle + \beta\langle w, v\rangle$

positive definite: $\langle u, u\rangle > 0$ whenever u is not the zero function.

The norm associated with the scalar product is denoted $\|u\| := \sqrt{\langle u, u\rangle}$.

The *eigenvalue problem* associated with the BVP (2.1) is

$$\mathcal{L}\phi + \lambda\phi = 0 \text{ in } \Omega, \quad \mathcal{B}\phi = 0 \text{ on } \partial\Omega \tag{2.6}$$

If this homogeneous problem admits a nontrivial solution ϕ for some constant λ, then ϕ is called an eigenfunction and λ is the associated eigenvalue.

Theorem 2.19 *The eigenvalues of a self adjoint BVP are real.*

PROOF. If λ is an eigenvalue and ϕ an associated eigenfunction, then the complex conjugate $\bar{\lambda}$ is an eigenvalue with eigenfunction $\bar{\phi}$, since the coefficients of (2.6) are real valued. Thus from Green's second identity for self adjoint BVPs we have

$$\begin{aligned} 0 &= \int_\Omega (\bar{\phi}\mathcal{L}\phi - \phi\mathcal{L}\bar{\phi})\,\mathrm{d}V \\ &= \langle \mathcal{L}\phi, \phi\rangle - \langle \mathcal{L}\bar{\phi}, \bar{\phi}\rangle \\ &= \langle -\lambda\phi, \phi\rangle - \langle -\bar{\lambda}\bar{\phi}, \bar{\phi}\rangle \\ &= -(\lambda - \bar{\lambda})\|\phi\|^2 \end{aligned}$$

and since $\|\phi\|^2 > 0$, we have $\lambda = \bar{\lambda}$, so the eigenvalues are real. ■

Theorem 2.20 *The eigenfunctions of a self-adjoint BVP are real and orthogonal on Ω.*

PROOF. Any real eigenvalue λ of problem (2.6) has a real eigenfunction, because if ϕ is an eigenfunction with nontrivial imaginary part $(\phi - \bar{\phi})/2$, then this imaginary part can be taken as an eigenfunction. For any two

eigenvalues λ_1 and λ_2 with associated real eigenfunctions ϕ_1 and ϕ_2, we have

$$\begin{aligned} 0 &= \int_\Omega (\phi_2 \mathcal{L}\phi_1 - \phi_1 \mathcal{L}\phi_2)\,\mathrm{d}V \\ &= \langle \phi_2, \mathcal{L}\phi_1\rangle - \langle \phi_1, \mathcal{L}\phi_2\rangle \\ &= \langle \phi_2, -\lambda_1\phi_1\rangle - \langle \phi_1, -\lambda_2\phi_2\rangle \\ &= -(\lambda_1 - \lambda_2)\langle \phi_1, \phi_2\rangle \end{aligned}$$

and if the eigenvalues are distinct then $\langle \phi_1, \phi_2\rangle = 0$, which is the orthogonality relation. If an eigenvalue has several linearly independent eigenfunctions then Gram-Schmidt orthogonalisation (described in linear algebra texts) can be applied to yield a mutually orthogonal set. ■

If λ is an eigenvalue of a self adjoint BVP with real eigenfunction ϕ, the energy formula gives the following formula known as the *Rayleigh quotient*:

$$\lambda = \frac{\int_\Omega (\phi_{\mathbf{x}}^T \mathbf{A}\phi_{\mathbf{x}} - c\phi^2)\,\mathrm{d}V - \int_{\partial\Omega} (\phi \mathbf{A}\phi_{\mathbf{x}})\cdot \mathbf{n}\,\mathrm{d}S}{\|\phi\|^2}$$

Using the Rayleigh quotient, the following results can be obtained.

Theorem 2.21 *If $c \le 0$, $fg \ge 0$, and f is not the zero function, the eigenvalues of the Robin problem are all positive. In particular, the eigenvalues of the Dirichlet problem with $c \le 0$ are positive.*

Theorem 2.22 *If $c \equiv 0$ then 0 is the smallest eigenvalue of the Neumann problem; a corresponding eigenfunction is the constant function $\phi \equiv 1$.*

Theorem 2.23 *If c is not the zero function and $c \le 0$, then the eigenvalues of the Neumann problem are positive.*

Example 5
The eigenvalue problem corresponding to the one-dimensional BVP of Example 4 is $\phi'' + \lambda\phi = 0$. Letting $\mu^2 := \lambda$, we solve this differential equation to get

```
> dsolve(diff(phi(x),x,x)+mu^2*phi(x)=0,phi(x)):
> phi:=unapply(rhs("),x);
```

$$\phi := x \to \_C1\cos(\mu\, x) + \_C2\sin(\mu\, x)$$

The homogeneous boundary conditions give a homogeneous linear system of equations in the parameters $\_C1$ and $\_C2$. The condition for this system to have a nontrivial solution is that the coefficient matrix be singular, that is, that it have zero determinant. This condition is called the *characteristic equation*.

```
> BC:=[phi(0)=0, phi(1)=0];
```

$$BC := [\_C1 = 0,\ \_C1\cos(\mu) + \_C2\sin(\mu) = 0]$$

```
> with(linalg):
> Ccoef:=genmatrix(BC,[_C1,_C2]);
```

$$Ccoef := \begin{bmatrix} 1 & 0 \\ \cos(\mu) & \sin(\mu) \end{bmatrix}$$

```
> CharEqn:=det(Ccoef)=0;
```

$$CharEqn := \sin(\mu) = 0$$

The roots of the characteristic equation are $\mu = j\pi$ with integer j. To find the corresponding eigenfunctions, we use the null space of the boundary condition's linear system's coefficient matrix.

```
> assume(j,integer): interface(showassumed=0);
> map(xi->simplify(subs(mu=j*Pi,xi)),Ccoef);
```

$$\begin{bmatrix} 1 & 0 \\ (-1)^j & 0 \end{bmatrix}$$

```
> NN:=nullspace(");
```

$$NN := \{[0,\ 1]\}$$

```
> subs(_C1=NN[1][1],_C2=NN[1][2],mu=j*Pi,phi(x));
```

$$\sin(j\,\pi\,x)$$

This is a nontrivial solution for nonzero values of j. The eigenvalues and eigenfunctions are therefore

```
> lambda:=j->j^2*Pi^2;
```

$$\lambda := j \to j^2\,\pi^2$$

```
> phi:=(j,x)->sin(j*Pi*x);
```

$$\phi := (j,\ x) \to \sin(j\,\pi\,x)$$

Verify the orthogonality of the eigenfunctions.

```
> assume(J,integer,K,integer);
> int(phi(J,x)*phi(K,x),x=0..1);
```

$$0$$

Verify Rayleigh's quotient formula.

```
> is(lambda(j)=int((diff(phi(j,x),x))^2,x=0..1)/
>     int(phi(j,x)^2,x=0..1));
```

true

2.5.2 Spectral Representation of Green's Function

In more advanced texts (e.g. [4, chap.11]) it is shown that the set of eigenvalues for the eigenvalue problem (2.6) is countably infinite and unbounded above, that is, the eigenvalues form a real sequence $\{\lambda_j\}$ with $\lim_{j\to\infty}\lambda_j = \infty$. It is also shown that the set of corresponding eigenfunctions $\{\phi_j\}$ is a complete basis for square-integrable functions on Ω. This means that any such function u can be expanded into the eigenfunction series (or *spectral*) representation

$$u(\mathbf{x}) = \sum_{j=1}^{\infty} \frac{\langle u, \phi_j\rangle}{\|\phi_j\|^2}\phi_j(\mathbf{x})$$

A spectral representation of the Green's function for the boundary value problem (2.1) would then have the form

$$G(\mathbf{x},\mathbf{x}') = \sum_j \psi_j(\mathbf{x}')\phi_j(\mathbf{x}) \tag{2.7}$$

where

$$\psi_j(\mathbf{x}') := \frac{\int_\Omega G(\mathbf{x},\mathbf{x}')\phi_j(\mathbf{x})\,\mathrm{d}V}{\|\phi_j\|^2} \tag{2.8}$$

An alternative formula for the ψ_j is found by substituting (2.7) into (2.2), which gives

$$\begin{aligned}\delta(\mathbf{x}-\mathbf{x}') &= \sum_j \psi_j(\mathbf{x}')\mathcal{L}\phi_j(\mathbf{x}) \\ &= -\sum_j \psi_j(\mathbf{x}')\lambda_j\phi_j(\mathbf{x})\end{aligned}$$

Taking the scalar product of both sides with ϕ_i gives

$$\begin{aligned}\int_\Omega \delta(\mathbf{x}-\mathbf{x}')\phi_i(\mathbf{x})\,\mathrm{d}V &= -\sum_j \psi_j(\mathbf{x}')\lambda_j\int_\Omega \phi_j(\mathbf{x})\phi_i(\mathbf{x})\,\mathrm{d}V \\ \phi_i(\mathbf{x}') &= -\sum_j \psi_j(\mathbf{x}')\lambda_j\delta_{ij}\|\phi_j\|^2 \\ &= -\psi_i(\mathbf{x}')\lambda_i\|\phi_i\|^2\end{aligned}$$

Then in place of (2.8) we have

$$\psi_i = -\frac{\phi_i}{\lambda_i\|\phi_i\|^2}$$

and substituting this into the Green's function series representation (2.7) gives

$$G(\mathbf{x}, \mathbf{x}') = -\sum_j \frac{\phi_j(\mathbf{x}')\phi_j(\mathbf{x})}{\lambda_j \|\phi_i\|^2} \tag{2.9}$$

It is evident from this spectral representation that the Green's function is symmetric (recall Theorem 2.14). Also, the representation is not valid for Neumann problems with $c \equiv 0$, which have a zero eigenvalue (and don't have a Green's function, as discussed on page 42).

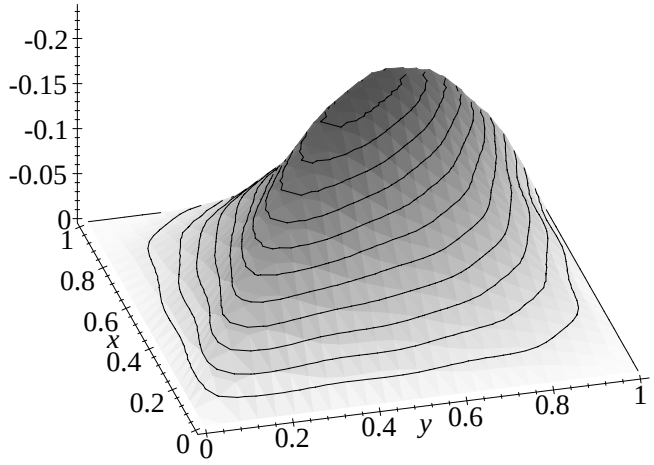
Example 5 (continued)
The Green's function for this example is assembled from the eigenvalues and eigenfunctions as follows.

```
>Gterm:=unapply(-phi(j,x)*phi(j,y)/lambda(j)
>    /int(phi(j,x)^2,x=0..1),(j,x,y)):
> G:=Sum(Gterm(j,x,y),j=1..infinity);
```

$$G := \sum_{j=1}^{\infty} \left(-2 \frac{\sin(j\,\pi\,x)\sin(j\,\pi\,y)}{j^2\,\pi^2} \right)$$

To plot it, use a finite number of terms of the series. The resulting plot resembles the one from Example 4 (page 40).

```
> Gapprox:=subs(infinity=6,G):
> plot3d(Gapprox,x=0..1,y=0..1,style=patchcontour,
>    axes=frame,shading=zgreyscale,orientation=[15,225]);
```



The solution of the BVP is given by the Green's function solution formula (Theorem 2.15) as

```
>u:=Sum(simplify(int(Gterm(j,x,y)*d(x),x=0..1)
>   +h[0]*D[2](Gterm)(j,0,y)-h[1]*D[2](Gterm)(j,1,y)),
>    j=1..infinity);
```

$$u := \sum_{j=1}^{\infty} \left(-2 \frac{\sin(j\,\pi\,y)\left(\int_0^1 \sin(j\,\pi\,x)\,\mathrm{d}(x)\,dx + h_0\,j\,\pi - h_1\,(-1)^j\,j\,\pi\right)}{j^2\,\pi^2} \right)$$

2.5.3 Separation of Variables

When the operator in the eigenvalue problem (2.6) is the laplacian $\mathcal{L} = \Delta$ in one of the standard orthogonal coordinate systems, and the domain Ω is suitably shaped, then the method of *separation of variables* can often be used. The following example demonstrates this technique.

Example 6
Consider the Poisson equation on a unit disk. The corresponding eigenvalue problem is

```
> with(linalg):
> EVP:=laplacian(phi(r,theta),[r,theta],coords=polar)
>         +mu^2*phi(r,theta)=0;
```

$$EVP := \frac{\left(\frac{\partial}{\partial r}\,\phi(r,\,\theta)\right) + r\,\left(\frac{\partial^2}{\partial r^2}\,\phi(r,\,\theta)\right) + \frac{\frac{\partial^2}{\partial \theta^2}\,\phi(r,\,\theta)}{r}}{r} + \mu^2\,\phi(r,\,\theta) = 0$$

Assume that the solution can be written as the product of two univariate functions, and rearrange to isolate the functions one each side of the equation.

```
> phi:=(r,theta)->R(r)*Theta(theta):
> expand(EVP);
```

$$\frac{\left(\frac{\partial}{\partial r}\,\mathrm{R}(r)\right)\,\Theta(\theta)}{r} + \left(\frac{\partial^2}{\partial r^2}\,\mathrm{R}(r)\right)\,\Theta(\theta) + \frac{\mathrm{R}(r)\,\left(\frac{\partial^2}{\partial \theta^2}\,\Theta(\theta)\right)}{r^2} + \mu^2\,\mathrm{R}(r)\,\Theta(\theta) = 0$$

```
> readlib(isolate):
> isolate(expand(lhs(EVP)*r^2/R(r)/Theta(theta))=0,r);
```

$$\frac{r\,\left(\left(\frac{\partial}{\partial r}\,\mathrm{R}(r)\right) + r\,\left(\frac{\partial^2}{\partial r^2}\,\mathrm{R}(r)\right) + r\,\mu^2\,\mathrm{R}(r)\right)}{\mathrm{R}(r)} = -\frac{\frac{\partial^2}{\partial \theta^2}\,\Theta(\theta)}{\Theta(\theta)}$$

Since the LHS (a function of r only) is equal to the RHS (a function of θ only), both sides are equal to a constant, call it σ^2. This gives two ODEs:

```
> Sep:=":
> ODE[1]:=lhs(Sep)*R(r)=sigma^2*R(r);
> ODE[2]:=rhs(Sep)*Theta(theta)=sigma^2*Theta(theta);
```

$$ODE_1 := r\left(\left(\frac{\partial}{\partial r}\,\mathrm{R}(r)\right) + r\left(\frac{\partial^2}{\partial r^2}\,\mathrm{R}(r)\right) + r\,\mu^2\,\mathrm{R}(r)\right) = \sigma^2\,\mathrm{R}(r)$$

$$ODE_2 := -\left(\frac{\partial^2}{\partial \theta^2}\,\Theta(\theta)\right) = \sigma^2\,\Theta(\theta)$$

The second ODE is a univariate eigenvalue problem similar to Example 5. Solving it gives

```
> dsolve(ODE[2],Theta(theta)):
> Theta:=unapply(rhs("),theta);
```

$$\Theta := \theta \to \_C1\cos(\sigma\,\theta) + \_C2\sin(\sigma\,\theta)$$

Since the PDE is on a disc the boundary conditions are periodic. These give the characteristic equation:

```
> BC:=[Theta(0)=Theta(2*Pi),
>      D(Theta)(0)=D(Theta)(2*Pi)]:
> Ccoef:=genmatrix(BC,[_C1,_C2]);
```

$$Ccoef := \begin{bmatrix} 1-\cos(2\,\sigma\,\pi) & -\sin(2\,\sigma\,\pi) \\ \sin(2\,\sigma\,\pi)\,\sigma & \sigma-\cos(2\,\sigma\,\pi)\,\sigma \end{bmatrix}$$

```
> CharEqn:=simplify(det(Ccoef))=0;
```

$$CharEqn := 2\,\sigma - 2\cos(2\,\sigma\,\pi)\,\sigma = 0$$

The characteristic equation has solution $\sigma = j$ with integer j, as the following calculation confirms:

```
> assume(j,integer):   interface(showassumed=0);
> is(simplify(subs(sigma=j,CharEqn)));
```

true

The eigenfunctions are found from the null space of the coefficient matrix.

```
> map(xi->simplify(subs(sigma=j,xi)),Ccoef);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

```
> NN:=nullspace(");
```

$$NN := \{[1,\ 0],\ [0,\ 1]\}$$

Each positive eigenvalue thus has two eigenfunctions, call them $T_1(j,\theta)$ and $T_2(j,\theta)$:

```
> T[1]:=unapply(subs(_C1=NN[1][1],_C2=NN[1][2],
>          sigma=j,Theta(theta)),(j,theta));
```

$$T_1 := (j,\ \theta) \rightarrow \cos(j\,\theta)$$

```
> T[2]:=unapply(subs(_C1=NN[2][1],_C2=NN[2][2],
>          sigma=j,Theta(theta)),(j,theta));
```

$$T_2 := (j,\ \theta) \rightarrow \sin(j\,\theta)$$

Gram-Schmidt orthogonalisation is not needed here since the eigenfunctions are already orthogonal, as the following calculations verify:

```
> assume(j,integer,k,integer);
> is(int(T[1](j,theta)*T[2](k,theta),theta=0..2*Pi)=0);
```

true

```
> assume(j,integer,k,integer,k  <>j);
> is(int(T[1](j,theta)*T[1](k,theta),theta=0..2*Pi)=0);
```

true

```
> is(int(T[2](j,theta)*T[2](k,theta),theta=0..2*Pi)=0);
```

true

Substituting $\sigma = j$ into the first ODE and solving gives

```
> dsolve(subs(sigma=j,ODE[1]),R(r));
```

$$\mathrm{R}(r) = \_C1\,\mathrm{BesselJ}(j,\ \mu\,r) + \_C2\,\mathrm{BesselY}(j,\ \mu\,r)$$

The Bessel function Y_j is unbounded at $r = 0$, so $\_C2 = 0$. The boundary condition $R(1) = 0$ gives the characteristic equation $J_j(\mu) = 0$. The root $\mu = 0$ gives a trivial solution, so only positive roots are chosen. Denoting the kth positive root as $\mu(j,k)$, the eigenfunctions for the disk problem are

```
> phi[1]:=unapply(BesselJ(j,mu(j,k)*r)*T[1](j,theta),
>          (j,k,r,theta));
```

$$\phi_1 := (j,\ k,\ r,\ \theta) \rightarrow \mathrm{BesselJ}(j,\ \mu(j,\ k)\,r)\cos(j\,\theta)$$

```
> phi[2]:=unapply(BesselJ(j,mu(j,k)*r)*T[2](j,theta),
>         (j,k,r,theta));
```

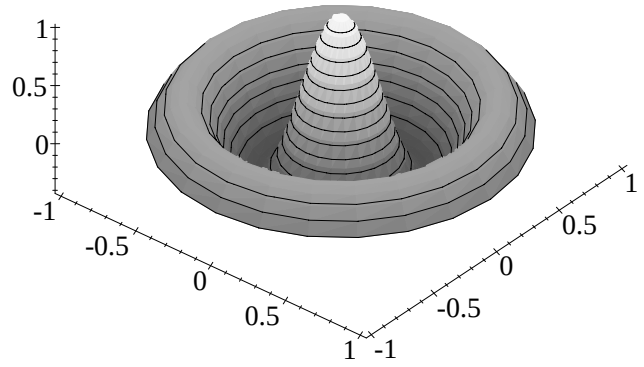
$$\phi_2 := (j,\ k,\ r,\ \theta) \rightarrow \mathrm{BesselJ}(j,\ \mu(j,\ k)\, r)\sin(j\,\theta)$$

To evaluate these functions, we need to find the zeros of the Bessel function J_j. The following Maple procedure uses the fact that the roots of Bessel functions interlace according to $z_{j-1,k} < z_{j,k} < z_{j-1,k+1}$ (where $z_{j,s}$ denotes the kth positive zero of J_j) to define the interval where `fsolve` searches for the root. Initial estimates for the roots of J_0 are provided by formulas from [1, Eqn 9.5.2].

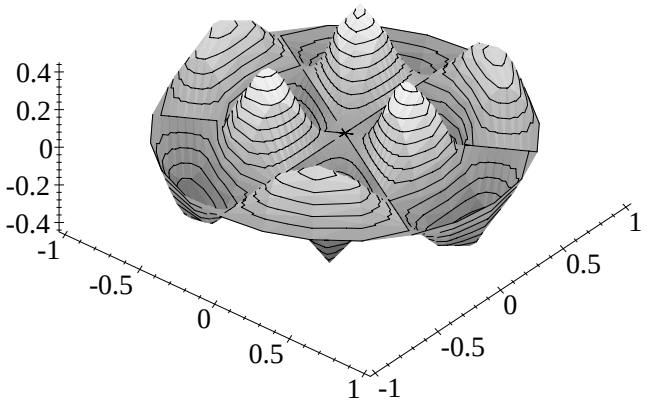
```
> mu:=proc(j,k)
>  local b,guess;
>  option remember;
>  if type(j,nonnegint) and type(k,posint) then
>    if j>0 then
>     fsolve(BesselJ(j,_Z),_Z,mu(j-1,k)..mu(j-1,k+1))
>    else
>     b:=(8*k-2)*Pi:
>     guess:=b/8+(1+(-4*31/3+32*3779/15/b^2)/b^2)/b:
>     fsolve(BesselJ(0,_Z),_Z,guess-1/100..guess+1/100)
>    fi:
>   else 'mu'(j,k):
>   fi:
>  end:
```

Let's plot a couple of eigenfunctions.

```
> readlib(addcoords)(z_cyl,[z,r,theta],[r*cos(theta),
>         r*sin(theta),z]):
> J:=0: K:=3: Mode:=1:
> plot3d(phi[Mode](J,K,r,theta),r=0..1,theta=0..2*Pi,
>   style=patchcontour,shading=zgreyscale,
>   coords=z_cyl,axes=frame,orientation=[-50,37]);
```




```
> J:=3: K:=2: Mode:=1:
> plot3d(phi[Mode](J,K,r,theta),r=0..1,theta=0..2*Pi,
>   style=patchcontour,shading=zgreyscale,
>   coords=z_cyl,axes=frame,orientation=[-50,37]);
```



Exercises

1. Verify that for any value of α, the function $u(x, y) = \alpha e^x \sin y$ solves the Dirichlet problem for $u_{xx} + u_{yy} = 0$ in the domain $(-\infty, \infty) \times (0, \pi)$ with boundary conditions $u(x, 0) = u(x, \pi) = 0$. Why is this not a counterexample to the uniqueness theorem?

2. Verify that the function $\epsilon \sin(m\pi x) \sin(m\pi y)/\sin m$ with integer m is a solution to the Dirichlet problem for $u_{xx} - u_{yy} = 0$ in the domain $(0, 1) \times (0, 1/\pi)$ with boundary conditions $u(x, 0) = u(0, y) = u(1, y) = 0, u(x, 1/\pi) = \epsilon \sin(m\pi x)$. Show that this solution does not depend continuously on the boundary data (*hint:* consider the points $(\frac{1}{2m}, \frac{1}{2m})$ for $m \geq 2$); why is this not a counterexample to Theorem 2.6?

3. Prove Theorems 2.8 and 2.9. Use Theorem 2.9 to show that the solution $u(x, y)$ of Example 2 is non-negative.

4. Show that the solution of $u_{xx} + u_{yy} = xy(x - \pi)(y - \pi)$ on $\Omega := (0, \pi) \times (0, \pi)$ with $u = 0$ on $\partial\Omega$ satisfies the inequality $u(x, y) \leq \frac{\pi^2}{2} \sin x \sin y$ on $\bar{\Omega}$.

5. Show that the ordinary differential equation

$$a(x)u''(x) + b(x)u'(x) + c(x)u(x) = d(x)$$

with $a > 0$ can be made formally self adjoint by multiplying through with a suitable smooth strictly positive function.

6. Show that the boundary data for the Neumann problem with $c \equiv 0$ has to satisfy the condition $\int_{\partial\Omega} h \, \mathrm{d}S = \int_\Omega d \, \mathrm{d}V$.

7. Show that the BVP $\Delta u = d$ on $(0, \pi) \times (0, \pi)$ with periodic boundary conditions

$$u(x, 0) = u(x, \pi) \; u_x(x, 0) = u_x(x, \pi) \quad (0 \leq x \leq \pi)$$
$$u(0, y) = u(\pi, y) \; u_y(0, y) = u_y(\pi, y) \quad (0 \leq y \leq \pi)$$

is self adjoint.

8. Prove Theorems 2.12 and 2.13.

9. Repeat Example 4 with the boundary value problem

$$u''(x) = d(x) \; (0 < x < 1), \quad u(0) = h_0, \; u'(1) = h_1$$

10. A rotation of the coordinate system corresponds to a change of variables of the form $\mathbf{x} = \mathbf{C}\mathbf{y}$, where $\mathbf{C}$ is a constant orthogonal matrix. Show that the laplacian operator is unchanged by a coordinate system rotation. A translation of the origin corresponds to a change of variables of the form $\mathbf{x} = \mathbf{y} + \mathbf{a}$ where $\mathbf{a}$ is a constant vector. Show that the laplacian operator is unchanged by a translation of the origin.

11. Compute the laplacian in paraboloidal coordinates

$$x_1 = y_1 y_2 \cos y_3, \quad x_2 = y_1 y_2 \sin y_3, \quad x_3 = \frac{1}{2}(y_1^2 - y_2^2)$$

using the techniques of chapter 1, and verify your answer using Maple's `laplacian` command.

12. Verify that the integrands in Poisson's integral formulas for the disc and the sphere are harmonic.

13. Verify that the regular part of the Green's function for the Poisson's equation Dirichlet problem on the half space $\Omega = \{(x_1, x_2, x_3), x_3 > 0\}$ is given by

$$H(\mathbf{x}') = \frac{1}{4\pi} \frac{1}{\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 + x_3')^2}}$$

and derive Poisson's integral formula for this problem.

14. Show that if u is harmonic in Ω then its value at a point $\mathbf{x}'$ is equal to its average over the volume of any ball $B_R \subset \Omega$ centred at $\mathbf{x}'$, that is,

$$u(\mathbf{x}') = \frac{1}{|B_R|} \int_{B_R} u(\mathbf{x} - \mathbf{x}') \, \mathrm{d}V$$

15. Show that if $u \in C^2(\Omega)$ has the mean value property then it is harmonic in Ω.

16. Show that the function $u := (x, y) \to xy(x^2 - y^2 + 2)$ is harmonic in R^2. Use this fact to find $\max_{[0,1]\times[0,1]} u$.

17. Solve the boundary value problem

$$u''(x) = \sin(x), \quad u(0) = u(1) = 0$$

using the Green's function from Example 4 and using the Green's function from Example 5.

18. Repeat Example 5 with the boundary value problem

$$u''(x) = d(x) \ (0 < x < 1), \quad u(0) = h_0, \ u'(1) = h_1$$

and compare with the answer from Exercise 9.

19. Let u be the solution of Poisson's equation on a disk with homogeneous Dirichlet boundary conditions and forcing function $d \equiv 1$. Find the value of u at the centre of the disk in two ways: first, using the formula for Green's function in section 2.4.2, and secondly, using the eigenfunctions in Example 6.

20. Find the Green's function for the Laplace equation Dirichlet problem on the square $(0, \alpha) \times (0, \beta)$.

Chapter 3

Parabolic PDEs

3.1 Initial-Boundary Value Problem

3.1.1 General Concepts

In this chapter we consider the linear nonisotropic diffusion-convection equation problem given by

$$\left.\begin{array}{rcll} \mathcal{L}u - u_t & = & d(\mathbf{x}, t) & \text{in } \Omega \times (0, \infty) \\ \mathcal{B}u & = & h(\mathbf{x}, t) & \text{on } \partial\Omega \times [0, \infty) \\ u(\mathbf{x}, 0) & = & k(\mathbf{x}) & (\mathbf{x} \in \bar{\Omega}) \end{array}\right\} \tag{3.1}$$

The notation is as in chapter 2, except that now u and the PDE coefficients are functions of both position $\mathbf{x}$ and time t. The diffusion-convection problem involves both boundary conditions and initial values. We are interested in the evolution of the solution as time moves forward from $t = 0$ into the future (Figure 3.1).

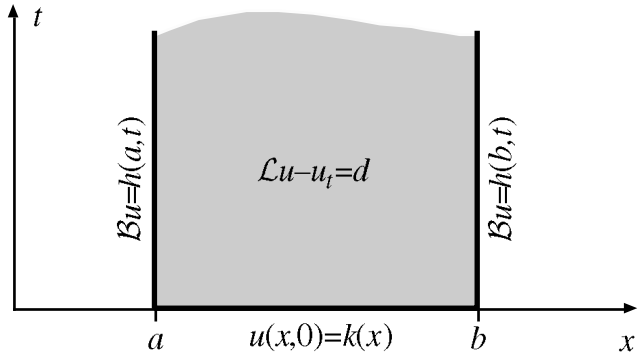


Figure 3.1: Domain for the diffusion-convection problem in one spatial dimension.

The PDE of the diffusion-convection problem is a linear second order equation. The PDE's principal part's coefficient matrix is singular, since the rows and columns corresponding to the variable t are zero. The diffusion-convection problem's PDE is therefore parabolic.

The elliptic BVP considered in chapter 2 can often be interpreted as the limiting case of a diffusion-convection problem that has settled down into a steady state, where $u_t = 0$. It is therefore not so surprising that many of the techniques for studying the diffusion-convection problem build on the results for the elliptic BVP. In the remainder of this section we use the maximum principle and the energy formula to derive results on uniqueness and continuous dependence on data.

3.1.2 Maximum Principle

Here is a Maximum Principle for the diffusion-convection problem.

Theorem 3.1 *Let $\mathcal{L}u - u_t \leq 0$ (respectively ≥ 0) with $c \leq 0$ in $\Omega \times (0, \infty)$, and let $T > 0$. If u has a negative minimum (resp. positive maximum) in $\bar{\Omega} \times [0, T]$ then this value is achieved at the initial time or on the boundary (and possibly elsewhere as well).*

PROOF. The following values are well defined, being minima of continuous functions on compact sets:

$$m := \min_{\bar{\Omega}\times[0,T]} u, \quad m_1 := \min_{\bar{\Omega}\times\{0\}} u, \quad m_2 := \min_{\partial\Omega\times[0,T]} u$$

We assume that the negative minimum $m < 0$ is not achieved at the initial time nor on the boundary, so that $m < \min\{m_1, m_2\} =: m_3$, and show that this leads to a contradiction. Introduce the auxiliary function v defined by the formula

$$v(\mathbf{x}, t) = u(\mathbf{x}, t) + \alpha(t - T) \quad (\mathbf{x} \in \bar{\Omega},\ t \in [0, T])$$

where α is a positive constant that is small enough that $m < m_3 - \alpha T$ and $m + \alpha T < 0$; for instance the value $\alpha = \frac{1}{2T} \min\{m_3 - m, -m\}$ will do. Since $v \leq u$ on $\bar{\Omega} \times [0, T]$, we have $m' := \min_{\bar{\Omega}\times[0,T]} v \leq m$. We also have the inequalities

$$v = u - \alpha T \geq m_1 - \alpha T \geq m_3 - \alpha T \quad \text{on } \bar{\Omega} \times \{0\}$$

and

$$v \geq u - \alpha T \geq m_2 - \alpha T \geq m_3 - \alpha T \quad \text{on } \partial\Omega \times [0, T]$$

so that $m_3 - \alpha T$ is a lower bound on values of v at the initial time and on the boundary. At a point $\mathbf{x}_0 \in \Omega$ and time $t_0 \in (0, T]$ where u achieves its minimum, we have

$$v = m + \alpha(t_0 - T) \leq m < m_3 - \alpha T$$

and so v cannot achieve its minimum at the initial time nor on the boundary. At a point $\mathbf{x}_1 \in \Omega$ and time $t_1 \in (0, T]$ where v does achieve its minimum,

$v_{\mathbf{xx}}$ is positive semidefinite (so that $\mathrm{tr}(\mathbf{A}v_{\mathbf{xx}}) \geq 0$), $v_{\mathbf{x}} = 0$, and we have

$$\begin{aligned} u_t &\geq \mathcal{L}u \\ &= \mathrm{tr}(\mathbf{A}u_{\mathbf{xx}}) + \mathbf{b}^T u_{\mathbf{x}} + cu \\ &= \mathrm{tr}(\mathbf{A}v_{\mathbf{xx}}) + \mathbf{b}^T v_{\mathbf{x}} + c(v - \alpha(t_1 - T)) \\ &= \mathrm{tr}(\mathbf{A}v_{\mathbf{xx}}) + c(m' - \alpha(t_1 - T)) \\ &\geq c(m' + \alpha T) \\ &\geq c(m + \alpha T) \\ &\geq 0 \end{aligned}$$

which in turn implies $v_t = u_t + \alpha > 0$, so that v is strictly increasing at the point $\mathbf{x}_1$ and time t_1 where it is supposed to achieve its minimum value. This contradiction proves the first part of the theorem. The proof of the second part of the theorem (i.e. the "respectively" part) follows by applying the first part to $-u$. ■

The above Maximum Principle can be used to show that the solution of the diffusion-convection problem with Dirichlet boundary conditions depends continuously on the boundary and initial data.

Theorem 3.2 *Let $\epsilon > 0$, and let $c \leq 0$ in $\Omega \times (0, \infty)$. If u is a solution of $\mathcal{L}u - u_t = d$ in $\Omega \times (0, \infty)$ with boundary condition $u = h_1$ on $\partial\Omega$ and initial condition $u = k_1$ at $t = 0$, and v solves the same PDE with $v = h_2$ on $\partial\Omega$ and $v = k_2$ at $t = 0$, with $\max_{\partial\Omega} |h_1 - h_2| \leq \epsilon$ for all $t > 0$ and $|k_1 - k_2| \leq \epsilon$ for all $\mathbf{x} \in \Omega$, then $|u - v| \leq \epsilon$ in $\Omega \times (0, \infty)$.*

PROOF. Because of the linearity of the PDE, the difference $w := u - v$ satisfies $\mathcal{L}w - w_t = 0$ in $\Omega \times (0, \infty)$ with boundary condition $w = h_1 - h_2$ on $\partial\Omega$ and initial condition $w = k_1 - k_2$ at $t = 0$. Assume that at some point $\mathbf{x}_0 \in \Omega$ and some time $T > 0$ we have $w(\mathbf{x}_0, T) > \epsilon$. Since $\max_{\bar{\Omega}\times[0,T]} w \geq w(\mathbf{x}_0, T) > \epsilon > 0$, this positive maximum is not achieved on the boundary (where $|w| = |h_1 - h_2| \leq \epsilon$) nor at the initial time (where $|w| = |k_1 - k_2| \leq \epsilon$). This contradicts Theorem 3.1, so we discharge the assumption and conclude that $w < \epsilon$ everywhere in $\Omega \times (0, \infty)$. The inequality $w > -\epsilon$ follows analogously. ■

Setting $h = h_1 = h_2$ and $k = k_1 = k_2$ in Theorem 3.2, gives the uniqueness theorem:

Theorem 3.3 *The solution of the diffusion-convection problem (3.1) with $c \leq 0$ and Dirichlet boundary conditions is unique.*

The monotonicity properties corresponding to Theorems 2.8–2.9 are summarised in

Theorem 3.4 *If $\mathcal{L}u - u_t \leq \mathcal{L}v - v_t$ and $c \leq 0$ in $\Omega \times (0, \infty)$, with $u \geq v$ on the boundary $\partial\Omega$ and at the initial time $t = 0$, then $u \geq v$ in $\Omega \times (0, \infty)$.*

The proof is similar to that of Theorem 3.2.

Example 1
This example uses the monotonicity property to derive a bound on the solution. Consider the one dimensional diffusion-convection problem

$$a(x,t)u_{xx} - u_t = \sin \pi x \quad (0 < x < 1,\ 0 < t)$$

with boundary and initial conditions

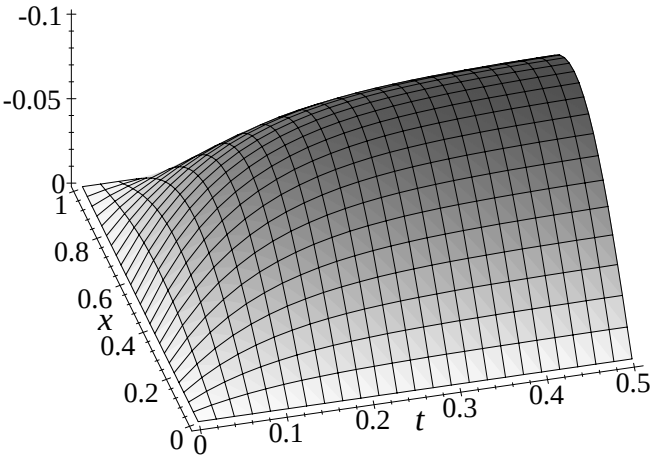
$$u(0,t) = u(1,t) = 0 \ \ (t \geq 0); \quad u(x,0) = 0 \ (0 \leq x \leq 1)$$

We show that if $0 < a \leq 1$ then the solution can be bounded by the inequality $u \leq v$, where

```
> v:=(x,t)->(exp(-Pi^2*t)-1)*sin(Pi*x)/(Pi^2);
```

$$v := (x,t) \mapsto \frac{\left(e^{-\pi^2 t} - 1\right)\sin(\pi\, x)}{\pi^2}$$

```
> plot3d(v(x,t),x=0..1,t=0..1/2,axes=frame,style=
>    patch,shading=zgreyscale,orientation=[15,215]);
```



The function v is the solution of the diffusion-convection problem with $a \equiv 1$, as the following calculations verify:

```
> is(diff(v(x,t),x,x)-diff(v(x,t),t)=sin(Pi*x));
```

true

```
> map(is, [ v(0,t)=0, v(1,t)=0, v(x,0)=0]);
```

[*true*, *true*, *true*]

If u satisfies the original problem then $w := u - v$ satisfies

$$a(x,t)w_{xx} - w_t = (1 - a(x,t))v_{xx}$$

with the same boundary and initial conditions. But $v_{xx} \geq 0$, as can be verified:

```
> assume(0<=x,x<=1,t>=0);
> is(diff(v(x,t),x,x)>=0);

                              true
```

The bound on the solution then follows from Theorem 3.4.

3.1.3 Uniqueness Results using the Energy Formula

We now restrict our attention to initial-boundary value problems (3.1) that are self-adjoint. This means that the elliptic operator is the formally self adjoint operator

$$\mathcal{L} := \nabla \cdot (\mathbf{A}u_{\mathbf{x}}) + cu$$

and the boundary condition operator annihilates the right hand side of Green's Second Identity (Theorem 2.3), that is,

$$\mathcal{B}u = 0, \mathcal{B}v = 0 \quad \Rightarrow \quad \int_\Omega v\mathcal{L}u - u\mathcal{L}v \, \mathrm{d}V = 0 \quad (t > 0)$$

Physically, this corresponds to a heat equation or a diffusion problem without convection.

The Dirichlet, Neumann, and Robin boundary conditions are now defined with the coefficients f, g, h considered to be functions of time t as well as position $\mathbf{x}$. These are self adjoint initial-boundary problems, since the proof of Theorem 2.4 goes through without changes. We have the following uniqueness result.

Theorem 3.5 *If $c \leq 0$ then the solution of the diffusion problem with Dirichlet, Neumann, or Robin boundary conditions (with $fg \geq 0$) is unique.*

PROOF. It suffices to consider the Robin boundary condition; the Dirichlet and Neumann conditions are special cases. We show that the only solution u of the associated homogeneous problem (in which $d = 0, h = 0, k = 0$) is the trivial one. The Energy Formula (Theorem 2.2) for u gives

$$0 = \int_\Omega u(\mathcal{L}u - u_t) \, \mathrm{d}V = \int_{\partial\Omega} (u\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n} \, \mathrm{d}S + \int_\Omega (-u_{\mathbf{x}}^T \mathbf{A}u_{\mathbf{x}} + cu^2 - uu_t) \, \mathrm{d}V \tag{3.2}$$

Partitioning the boundary into disjoint sets $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $f \neq 0$ on Γ_1 and $g \neq 0$ on Γ_2, and introducing the energy integral

$$E(t) := \frac{1}{2} \int_\Omega u^2 \, \mathrm{d}V$$

equation (3.2) can be written

$$\begin{aligned}\dot{E}(t) &= \int_\Omega u u_t \, \mathrm{d}V \\ &= \int_{\Gamma_1} \frac{-g}{f}[(\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n}]^2 \, \mathrm{d}S + \int_{\Gamma_2} \frac{-f}{g} u^2 \, \mathrm{d}S + \int_\Omega (-u_{\mathbf{x}}^T \mathbf{A} u_{\mathbf{x}} + cu^2) \, \mathrm{d}V\end{aligned}$$

Now $E(0) = 0$, $E(t) \geq 0$ $(t \geq 0)$ and $\dot{E}(t) \leq 0$ $(t \geq 0)$ together imply that $E(t) = 0$ $(t \geq 0)$. This forces u to be the trivial solution. ■

Notice how the diffusion problem with Neumann boundary conditions does not need to be treated separately like the Neumann boundary value problem, whose solution was only unique up to an additive constant.

3.2 Solution Techniques

3.2.1 System Concepts

The diffusion problem can be treated as a *dynamical system* with state u, inputs d and h, and initial state k. The system is

causal: inputs $\{d(\mathbf{x}, t), h(\mathbf{x}, t)\}$ in the "future" $t > t'$ do not affect the solution $u(\mathbf{x}, t')$ (Exercise 4);

infinite-dimensional: the state space is $C(\bar{\Omega}) \cap C^2(\Omega)$;

linear: if inputs $\{d_1, h_1\}$ and initial state k_1 give solution u_1, while inputs $\{d_2, h_2\}$ and initial state k_2 give solution u_2, then inputs $\{\alpha d_1 + \beta d_2, \alpha h_1 + \beta h_2\}$ and initial state $\alpha k_1 + \beta k_2$ will give solution $\alpha u_1 + \beta u_2$.

If the PDE operator $\mathcal{L}$ and boundary operator $\mathcal{B}$ are independent of time t, then the system is *time invariant*, and the shape of the solution is independent of a shift of the origin of the time axis. To make this statement more explicit, let H represent the Heaviside unit step function, and let $\mathcal{D}_\tau$ represent the ideal delay operator

$$\mathcal{D}_\tau u(\mathbf{x}, t) = u(\mathbf{x}, t - \tau)\mathrm{H}(t - \tau) \quad (t \geq 0)$$

for any fixed $\tau \geq 0$ (see Figure 3.2). Time invariance then means that if u is the solution of the diffusion problem (3.1) then $v := \mathcal{D}_\tau u$ is the solution of the same problem with time origin translated to τ:

$$\begin{aligned} \mathcal{L}v - v_t &= \mathcal{D}_\tau d \quad \text{in } \Omega \times (\tau, \infty) \\ \mathcal{B}v &= \mathcal{D}_\tau h \quad \text{on } \partial\Omega \times [\tau, \infty) \\ v(\mathbf{x}, \tau) &= k(\mathbf{x}) \quad (\mathbf{x} \in \bar{\Omega}) \end{aligned}$$

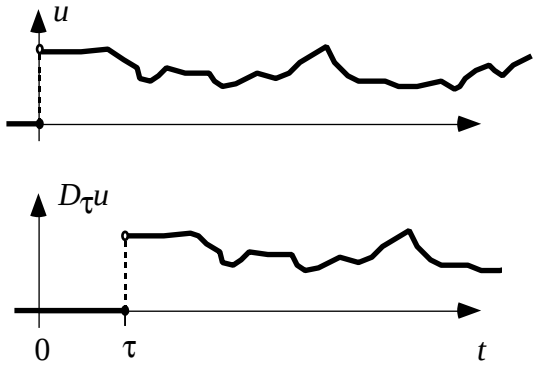


Figure 3.2: The delay operator $\mathcal{D}_\tau$.

3.2.2 Duhamel's Principle

Duhamel's Principle is a name given to a variety of formulas that represent the solution of a linear time invariant system as a superposition of solutions of simple test problems.

One such result is the following, in which the response of the system to a constant "step" input is used to calculate the response to any time varying input.

Theorem 3.6 *The solution of the diffusion problem (3.1) with time invariant operators $\mathcal{L}$, $\mathcal{B}$ is given by*

$$u(\mathbf{x},t) = \frac{\partial}{\partial t}\int_0^t v(\mathbf{x},t-\tau,\tau)\,\mathrm{d}\tau$$

where, for every fixed $\tau \geq 0$, $v(\mathbf{x},t,\tau)$ is the solution of the problem with constant inputs

$$\begin{aligned}
\left(\mathcal{L}-\tfrac{\partial}{\partial t}\right)v(\mathbf{x},t,\tau) &= \mathrm{H}(t)d(\mathbf{x},\tau) \quad (\mathbf{x}\in\Omega, t>0)\\
\mathcal{B}v(\mathbf{x},t,\tau) &= \mathrm{H}(t)h(\mathbf{x},\tau) \quad (\mathbf{x}\in\partial\Omega, t\geq 0)\\
v(\mathbf{x},0,\tau) &= k(\mathbf{x}) \quad (\mathbf{x}\in\bar{\Omega})
\end{aligned}$$

PROOF. First, verify that the proposed formula for the solution satisfies the initial condition.

```
> diff(int(v(x,t-tau,tau),tau=0..t),t);
```

$$\int_0^t D_2(v)(x,t-\tau,\tau)d\tau + v(x,0,t)$$

```
> subs(v(x,0,t)=k(x),");
```

$$\int_0^t D_2(v)(x,t-\tau,\tau)d\tau + k(x)$$

```
> u:=unapply(",x,t):
> is(u(x,0)=k(x));
```

true

Next, verify that the formula satisfies the boundary condition.

$$\begin{aligned}
\mathcal{B}u(\mathbf{x},t) &= \frac{\partial}{\partial t}\int_0^t \mathcal{B}v(\mathbf{x},t-\tau,\tau)\,\mathrm{d}\tau\\
&= \frac{\partial}{\partial t}\int_0^t \mathrm{H}(t-\tau)h(\tau)\,\mathrm{d}\tau\\
&= \int_0^t \delta(t-\tau)h(\tau)\,\mathrm{d}\tau\\
&= h(t)
\end{aligned}$$

Finally, verify that the formula satisfies the PDE.

$$\mathcal{L}u(\mathbf{x},t) = \frac{\partial}{\partial t}\int_0^t \mathcal{L}v(\mathbf{x},t-\tau,\tau)\,\mathrm{d}\tau$$

$$\begin{aligned} &= \frac{\partial}{\partial t}\int_0^t [\mathrm{H}(t-\tau)d(\mathbf{x},\tau) + v_t(\mathbf{x}, t-\tau, \tau)]\,\mathrm{d}\tau \\ &= \int_0^t \delta(t-\tau)d(\mathbf{x},\tau)\,\mathrm{d}\tau + \frac{\partial}{\partial t}[u(\mathbf{x},t) - k(\mathbf{x})] \\ &= d(\mathbf{x},t) + u_t(\mathbf{x},t) \end{aligned}$$

This concludes the proof. ■

The proof of Theorem 3.6 did not use any special properties of the diffusion equation other than its properties as a causal linear time invariant system. Therefore, formulas analogous to Theorem 3.6 hold for any such system. For instance, in Chapter 4 we'll give Duhamel Principles for the wave equation.

The step response v needed in the Duhamel Principle could be found by any method, or could even be measured data from a physical experiment. No matter how it is obtained, once the step response is known, the response for arbitrary input can be derived. This is demonstrated in the following example.

Example 2
Consider the one dimensional heat equation $v_{xx} - v_t = 0$ with boundary conditions $v(0,\, t) = 0$, $v(1,\, t) = h_1(\tau)$ and zero initial conditions. The following step response is derived later in Example 4 (see also Exercise 7):

```
> vterm:=unapply((2/Pi/n)*(-1)^n*exp(-n^2*Pi^2*t)
>       *sin(n*Pi*x),(n,x,t));
```

$$vterm := (n,\, x,\, t) \to 2\,\frac{(-1)^n\, e^{(-n^2\,\pi^2\,t)}\sin(n\,\pi\,x)}{\pi\,n}$$

```
> v:=unapply((x+Sum(vterm(n,x,t),n=1..infi nity))
>       *h[1](tau),(x,t,tau));
```

$$v := (x,\, t,\, \tau) \to \left(x + \left(\sum_{n=1}^{\infty}\left(2\,\frac{(-1)^n\, e^{(-n^2\,\pi^2\,t)}\sin(n\,\pi\,x)}{\pi\,n}\right)\right)\right) h_1(\tau)$$

Verify that it satisfies the PDE and the boundary conditions.

```
> is(diff(v(x,t,tau),x,x)-diff(v(x,t,tau),t)=0);
```

true

```
> assume(n,integer);  interface(showassumed=0);
> is(value(v(1,t,tau)=h[1](tau)));
```

true

```
> is(value(v(0,t,tau))=0);
```

$$true$$

When we try to verify that it satisfies the initial condition, we get

```
> k:=v(x,0,tau);
```

$$k := \left(x + 2\,\frac{\sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\,\pi\,x)}{n}}{\pi} \right) \mathrm{h}_1(\tau)$$

This cancels to zero when we use the formula

$$x = \sum_{n=1}^{\infty} 2 \int_0^1 \xi \sin(n\,\pi\,\xi)\,d\xi \sin(n\,\pi\,x)$$

for the Fourier sine series of x:

```
> FourierSeries:=x=Sum(2*int(xi*sin(n*Pi*xi),
>             xi=0..1)*sin(n*Pi*x),n=1..infi nity);
```

$$FourierSeries := x = \sum_{n=1}^{\infty} \left(-2\,\frac{(-1)^n \sin(n\,\pi\,x)}{n\,\pi} \right)$$

```
> is(simplify(k,{FourierSeries})=0);
```

$$true$$

Now let's try to solve the problem with the specific time varying input $h_1(t) = \sin(t)$. Applying Duhamel's Principle gives

```
> h[1]:=sin:
> diff(int(v(x,t-tau,tau),tau=0..t),t);
```

$$\int_0^t 2\,\frac{\left(\sum_{n=1}^{\infty} \left(-(-1)^n\,n\,\pi^2\,e^{(-n^2\,\pi^2\,(t-\tau))} \sin(n\,\pi\,x) \right) \right) \sin(\tau)}{\pi}\,d\tau + \left(x + 2\,\frac{\sum_{n=1}^{\infty} \frac{(-1)^n\,e^0 \sin(n\,\pi\,x)}{n}}{\pi} \right) \sin(t)$$

Unfortunately it seems that Maple does not interchange the integration and summation operators. We get better results by rewriting the Duhamel formula with the integrals on the inside of the summations.

```
> u:=unapply(diff(int(x*h[1](tau),tau=0..t)+Sum(
>     int(h[1](tau)*vterm(n,x,t-tau),tau=0..t),
>      n=1..infinity),t),(x,t));
```

$$u := (x,\ t) \quad \rightarrow \quad \sin(t)\,x + \sum_{n=1}^{\infty} \left(2\,\frac{(\sin(t) + n^2\,\pi^2\cos(t))\,(-1)^n \sin(n\,\pi\,x)}{(n^4\,\pi^4 + 1)\,\pi\,n} - 2\,\frac{n\,\pi\,e^{(-n^2\,\pi^2\,t)}\,(-1)^n \sin(n\,\pi\,x)}{n^4\,\pi^4 + 1} \right)$$

Verify that u satisfies the PDE

```
> combine(diff(u(x,t),x,x)-diff(u(x,t),t));
```

$$\left(\sum_{n=1}^{\infty} \left(-\frac{(\sin(n\,\pi\,x + t) + \sin(n\,\pi\,x - t))\,(-1)^n}{n\,\pi} \right) \right) - \cos(t)\,x$$

```
> is(simplify(",{FourierSeries})=0);
```

true

the boundary conditions

```
> is(value(u(1,t)=h[1](t))) and is(value(u(0,t))=0);
```

true

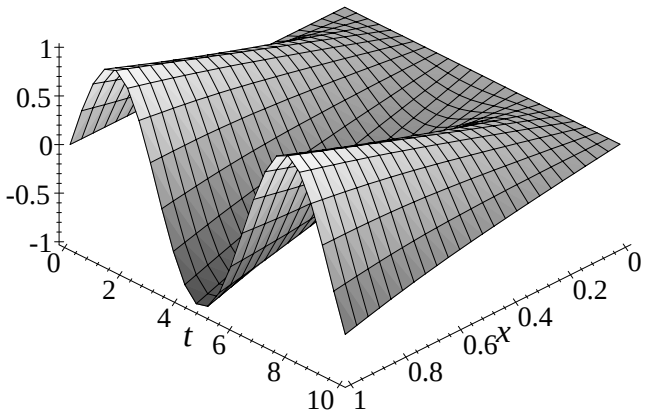
and the initial condition

```
> is(value(u(x,0))=0);
```

true

Finally, plot the solution, using a finite number of terms of the sum.

```
> U3:=subs(infinity=3,u(x,t)):
> plot3d(U3,x=0..1,t=0..10,axes=frame,style=
>          patch,shading=zgreyscale);
```



Another version of Duhamel's Principle is the following, in which the solution is represented as the superposition of a set of "free response" problems.

Theorem 3.7 *The solution of the diffusion problem*

$$\begin{aligned} \left(\mathcal{L} - \tfrac{\partial}{\partial t}\right) u(\mathbf{x}, t) &= d(\mathbf{x}, t) \quad (\mathbf{x} \in \Omega, t > 0) \\ \mathcal{B}u(\mathbf{x}, t) &= 0 \quad (\mathbf{x} \in \partial\Omega, t > 0) \\ u(\mathbf{x}, 0) &= 0 \quad (\mathbf{x} \in \Omega) \end{aligned}$$

with time invariant operators $\mathcal{L}$, $\mathcal{B}$ is given by

$$u(\mathbf{x}, t) = \int_0^t v(\mathbf{x}, t - \tau, \tau)\, \mathrm{d}\tau$$

where, for every fixed $\tau \geq 0$, $v(\mathbf{x}, t, \tau)$ is the solution of the problem

$$\begin{aligned} \left(\mathcal{L} - \tfrac{\partial}{\partial t}\right) v(\mathbf{x}, t, \tau) &= 0 \quad (\mathbf{x} \in \Omega, t > 0) \\ \mathcal{B}v(\mathbf{x}, t, \tau) &= 0 \quad (\mathbf{x} \in \partial\Omega, t > 0) \\ v(\mathbf{x}, 0, \tau) &= -d(\mathbf{x}, \tau) \quad (\mathbf{x} \in \Omega) \end{aligned}$$

The proof is similar to that of Theorem 3.6. (Exercise 5).

3.2.3 Green's Functions via Laplace Transforms

The Laplace transform is convenient for studying initial value problems for linear time invariant dynamical systems. The Laplace transform of $u(\mathbf{x}, t)$, denoted $\hat{u}(\mathbf{x}, s)$, has the following properties.

Linearity: $\widehat{u + v} = \hat{u} + \hat{v}$, $\widehat{\alpha u} = \alpha \hat{u}$;

Derivative: If $v = u_t$ then $\hat{v} = s\hat{u} - u(\mathbf{x}, 0)$;

Product: $\hat{u}\hat{v} = \widehat{u * v}$, where $u * v$ denotes the convolution, defined by $(u * v)(\mathbf{x}, t') = \int_0^{t'} u(\mathbf{x}, t' - t) v(\mathbf{x}, t)\, \mathrm{d}t$.

Taking the Laplace transform of the initial-boundary value problem (3.1) with time invariant operators gives

$$(\mathcal{L} - s)\hat{u}(\mathbf{x}, s) = \hat{d}(\mathbf{x}, s) - k(\mathbf{x}) \text{ in } \Omega, \quad \mathcal{B}\hat{u}(\mathbf{x}, s) = \hat{h}(\mathbf{x}, s) \text{ on } \partial\Omega \quad (3.3)$$

where the Laplace variable s is treated as a symbolic parameter. Notice that the initial condition is now incorporated into the PDE, and (3.3) has the form of a boundary value problem. We therefore apply the techniques of the previous chapter.

We first find the Laplace transform of the singularity function, $\hat{F}(\mathbf{x}, \mathbf{x}', s)$, as a general solution of the PDE

$$(\mathcal{L} - s)\hat{F}(\mathbf{x}, \mathbf{x}', s) = \delta(\mathbf{x} - \mathbf{x}')$$

Then we find a regular solution $\hat{H}(\mathbf{x}, \mathbf{x}', s)$ to the BVP

$$(\mathcal{L} - s)\hat{H} = 0 \text{ in } \Omega, \ \mathcal{B}\hat{H} = -\mathcal{B}\hat{F} \text{ on } \partial\Omega$$

Finally, the Laplace transform of the Green's function is defined as $\hat{G} := \hat{F} + \hat{H}$.

Applying Theorem 2.15, we find the solution of the self-adjoint diffusion problem with Robin boundary condition in the form

$$\begin{aligned} \hat{u}(\mathbf{x}', s) &= \int_\Omega \hat{G}(\mathbf{x}, \mathbf{x}', s)(\hat{d}(\mathbf{x}, s) - k(\mathbf{x}))\,\mathrm{d}V \\ &\quad + \int_{\Gamma_1} \frac{\hat{h}(\mathbf{x}, s))}{f(\mathbf{x})} \left(\mathbf{A}(\mathbf{x})\hat{G}_{\mathbf{x}}(\mathbf{x}, \mathbf{x}', s)\right) \cdot \mathbf{n}\,\mathrm{d}S \\ &\quad - \int_{\Gamma_2} \hat{G}(\mathbf{x}, \mathbf{x}', s)\frac{\hat{h}(\mathbf{x}, s)}{g(\mathbf{x})}\,\mathrm{d}S \end{aligned}$$

Taking the inverse Laplace transform of both sides gives

Theorem 3.8 *The solution of the time invariant self-adjoint diffusion problem with Robin boundary conditions is*

$$\begin{aligned} u(\mathbf{x}', t') &= \int_0^{t'} \int_\Omega G(\mathbf{x}, \mathbf{x}', t' - t)d(\mathbf{x}, t)\,\mathrm{d}V\,\mathrm{d}t - \int_\Omega G(\mathbf{x}, \mathbf{x}', t')k(\mathbf{x})\,\mathrm{d}V \\ &\quad + \int_0^{t'} \int_{\Gamma_1} \frac{h(\mathbf{x}, t))}{f(\mathbf{x})} \left(\mathbf{A}(\mathbf{x})G_{\mathbf{x}}(\mathbf{x}, \mathbf{x}', t' - t)\right) \cdot \mathbf{n}\,\mathrm{d}S\,\mathrm{d}t \\ &\quad - \int_0^{t'} \int_{\Gamma_2} G(\mathbf{x}, \mathbf{x}', t' - t)\frac{h(\mathbf{x}, t)}{g(\mathbf{x})}\,\mathrm{d}S\,\mathrm{d}t \end{aligned}$$

As special cases of this, the solution of the Dirichlet problem is

$$\begin{aligned} u(\mathbf{x}', t') &= \int_0^{t'} \int_\Omega G(\mathbf{x}, \mathbf{x}', t' - t)d(\mathbf{x}, t)\,\mathrm{d}V\,\mathrm{d}t - \int_\Omega G(\mathbf{x}, \mathbf{x}', t')k(\mathbf{x})\,\mathrm{d}V \\ &\quad + \int_0^{t'} \int_{\partial\Omega} h(\mathbf{x}, t) \left(\mathbf{A}G_{\mathbf{x}}(\mathbf{x}, \mathbf{x}', t' - t)\right) \cdot \mathbf{n}\,\mathrm{d}S\,\mathrm{d}t \end{aligned}$$

and the solution of the Neumann problem is

$$\begin{aligned} u(\mathbf{x}', t') &= \int_0^{t'} \int_\Omega G(\mathbf{x}, \mathbf{x}', t' - t)d(\mathbf{x}, t)\,\mathrm{d}V\,\mathrm{d}t - \int_\Omega G(\mathbf{x}, \mathbf{x}', t')k(\mathbf{x})\,\mathrm{d}V \\ &\quad - \int_0^{t'} \int_{\partial\Omega} G(\mathbf{x}, \mathbf{x}', t' - t)h(\mathbf{x}, t)\,\mathrm{d}S\,\mathrm{d}t \end{aligned}$$

Notice that the Green's function will be symmetric, in the sense that $G(\mathbf{x}, \mathbf{x}', t) = G(\mathbf{x}', \mathbf{x}, t)$, since the Laplace transformed problem is a self adjoint boundary value problem. Notice also that Neumann problems do not require special treatment (such as Neumann functions).

Example 3
Consider the one dimensional heat equation

$$u_{xx} - u_t = d(x,t)$$

on $(0,1) \times (0,\infty)$, with Dirichlet boundary conditions

$$u(0,t) = h_0(t),\ u(1,t) = h_1(t)$$

and initial condition

$$u(x,t) = k(x)$$

First we find a singularity function.

```
> dsolve(diff(u(x),x,x)-s*u(x)=Dirac(x-y),u(x));
```

$$\mathrm{u}(x) = -\frac{1}{2}\,\frac{\mathrm{Heaviside}(x-y)\,(-1+e^{(-2\sqrt{s}\,(x-y))})\,e^{(\sqrt{s}\,(x-y))}}{\sqrt{s}} + \_C1\,e^{(\sqrt{s}\,x)} + \_C2\,e^{(-\sqrt{s}\,x)}$$

```
> Fhat:=unapply(subs(_C1=0,_C2=0,rhs(")),(x,y));
```

$$Fhat := (x,\ y) \to -\frac{1}{2}\,\frac{\mathrm{Heaviside}(x-y)\,(-1+e^{(-2\sqrt{s}\,(x-y))})\,e^{(\sqrt{s}\,(x-y))}}{\sqrt{s}}$$

Next, find the regular part of the Green's function.

```
>  assume(0<Y,Y<1);  interface(showassumed=0);
> dsolve({diff(u(x),x,x)-s*u(x)=0,u(0)=-Fhat(0,Y),
>      u(1)=-Fhat(1,Y)},u(x));
```

$$\mathrm{u}(x) = \frac{1}{2}\,\frac{(-1+(e^{(\sqrt{s}\,Y-\sqrt{s})})^2)\,e^{(2\sqrt{s}-\sqrt{s}\,Y)}\,e^{(\sqrt{s}\,x)}}{\sqrt{s}\,((e^{(\sqrt{s})})^2-1)} - \frac{1}{2}\,\frac{(-1+(e^{(\sqrt{s}\,Y-\sqrt{s})})^2)\,e^{(2\sqrt{s}-\sqrt{s}\,Y)}\,e^{(-\sqrt{s}\,x)}}{\sqrt{s}\,((e^{(\sqrt{s})})^2-1)}$$

```
> Hhat:=unapply(simplify(subs(Y=y,rhs(("))),(x,y));
```

$$Hhat := (x,\ y) \to -\frac{1}{2}\,\frac{(-1+e^{(2\sqrt{s}\,(y-1))})\,(-e^{(\sqrt{s}\,(2-y+x))}+e^{(-\sqrt{s}\,(-2+y+x))})}{\sqrt{s}\,(e^{(2\sqrt{s})}-1)}$$

Assemble the two parts of the Green's function and verify symmetry, considering the cases $x < y$ and $x > y$ separately.

```
> Ghat:=unapply(Fhat(x,y)+Hhat(x,y),(x,y)):
> assume(X<Y);  simplify(Ghat(X,Y)-Ghat(Y,X));
```

$$0$$

```
> assume(Y<X);simplify(Ghat(X,Y)-Ghat(Y,X));

                                0
```

The inverse Laplace transform of $\hat{G}$ can then be found with the aid of tables (Exercise 7).

The convolution formulas appearing in Theorem 3.8 resemble the Duhamel Principle formulas, but are not quite the same. The Green's function $G(\mathbf{x}, \mathbf{x}', t)$ has a physical interpretation as the response at a point $\mathbf{x}'$ and time t due to a unit impulsive input applied at a point $\mathbf{x}$ at the time $t = 0$.

3.2.4 Method of Eigenfunction Expansion

In the previous section we changed the initial-boundary value problem into a pure boundary value problem, using the Laplace transform to change the time derivative into a parameter. Here we take the opposite approach, changing the initial-boundary value problem into a set of pure initial-value ODE problems. This is done using the eigenvalues and eigenfunctions of the associated boundary value problem. For convenience, we assume that $\mathcal{L}$ and $\mathcal{B}$ are time-invariant operators.

Theorem 3.9 *The Green's function for the time invariant self-adjoint diffusion problem (3.1) is given by*

$$G(\mathbf{x}, \mathbf{x}', t) = -\sum_{j=1}^{\infty} \frac{\phi_j(\mathbf{x})\phi_j(\mathbf{x}')}{\|\phi_j\|^2} e^{-\lambda_j t}$$

where $\{(\lambda_j, \phi_j) : j = 1 \ldots \infty\}$ *is the set of eigenvalues and eigenfunctions from*

$$\mathcal{L}\phi + \lambda\phi = 0, \quad \mathcal{B}\phi = 0$$

PROOF. Substituting the eigenfunction expansion

$$u(\mathbf{x}, t) = \sum_{j=1}^{\infty} U_j(t)\phi_j(\mathbf{x}) \tag{3.4}$$

into the diffusion PDE with no inputs gives

$$\begin{aligned} 0 &= \sum_j \left[U_j(t)\mathcal{L}\phi_j(\mathbf{x}) - \dot{U}_j(t)\phi_j(\mathbf{x}) \right] \\ &= -\sum_j \left[\lambda_j U_j(t) + \dot{U}_j(t) \right] \phi_j(\mathbf{x}) \end{aligned}$$

Taking the scalar product of this with ϕ_i gives

$$\begin{aligned} 0 &= -\sum_j \left[\lambda_j U_j(t) + \dot{U}_j(t) \right] \langle \phi_j, \phi_i \rangle \\ &= -\left[\dot{U}_j(t) + \lambda_j U_j(t) \right] \|\phi_j\|^2 \end{aligned}$$

This simplifies to the decoupled system of ODEs

$$\dot{U}_j(t) + \lambda_j U_j(t) = 0 \quad (j = 1 \ldots \infty)$$

whose solutions are

$$U_j(t) = U_j(0)e^{-\lambda_j t} \quad (j = 1 \ldots \infty) \tag{3.5}$$

Substituting $t = 0$ into (3.4) gives the initial values of the diffusion problem as

$$k(\mathbf{x}) = u(\mathbf{x}, 0) = \sum_{j=1}^{\infty} U_j(0)\phi_j(\mathbf{x})$$

Again, we take the scalar product of this with ϕ_i and using orthogonality. This gives us the ODE initial values

$$U_j(0) = \frac{\langle k, \phi_j \rangle}{\|\phi_j\|^2} \quad (j = 1 \ldots \infty)$$

Substituting this and (3.5) into (3.4) gives the solution of the diffusion PDE with no inputs in the form

$$\begin{aligned} u(\mathbf{x}', t) &= \sum_{j=1}^{\infty} U_j(0)e^{-\lambda_i t}\phi_j(\mathbf{x}') \\ &= \sum_{j=1}^{\infty} \frac{\langle k, \phi_j \rangle}{\|\phi_j\|^2} e^{-\lambda_i t}\phi_j(\mathbf{x}') \\ &= \int_{\Omega} \left[\sum_{j=1}^{\infty} \frac{\phi_j(\mathbf{x})\phi_j(\mathbf{x}')}{\|\phi_j\|^2} e^{-\lambda_i t} \right] k(\mathbf{x})\, \mathrm{d}V \end{aligned}$$

Comparing this with the Green's function formula (Theorem 3.8) for the solution of the same problem, namely,

$$u(\mathbf{x}, t) = -\int_{\Omega} G(\mathbf{x}, \mathbf{x}', t)k(\mathbf{x})\, \mathrm{d}V$$

gives the Green's function formula that was asserted. ■

Example 4

From Example 2.5 we know that the eigenvalues and eigenfunctions for the Dirichlet problem $u_{xx} = d$ on $[0, 1]$ are

```
> lambda:=j->j^2*Pi^2;
```

$$\lambda := j \to j^2 \pi^2$$

```
> phi:=(j,x)->sin(j*Pi*x);
```

$$\phi := (j,\ x) \to \sin(j\,\pi\,x)$$

The Green's function can therefore be written

```
> assume(j,posint):    interface(showassumed=0):
> Gterm:=unapply(-phi(j,x)*phi(j,y)*exp(-lambda(j)*t)
>   /int(phi(j,x)^2,x=0..1),(j,x,y,t)):
> G:=Sum(Gterm(j,x,y,t),j=1..infinity);
```

$$G := \sum_{j=1}^{\infty} \left(-2 \sin(j\,\pi\,x) \sin(j\,\pi\,y)\, e^{(-j^2\,\pi^2\,t)}\right)$$

In Example 3.2 we needed the solution of the heat equation when a step input is applied at the right boundary. From the Green's function solution formula we have

```
> vterm:=int(D[2](Gterm)(j,1,x,t-tau),tau=0..t);
```

$$vterm := -2\,\frac{(-1)^j \sin(j\,\pi\,x)}{j\,\pi} + 2\,\frac{e^{(-j^2\pi^2 t)}(-1)^j \sin(j\,\pi\,x)}{j\,\pi}$$

```
> vv:=sum(Sum(op(n,vterm),j=1..infinity),n=1..2);
```

$$vv := \left(\sum_{j=1}^{\infty} \left(-2\,\frac{(-1)^j \sin(j\,\pi\,x)}{j\,\pi}\right)\right) + \left(\sum_{j=1}^{\infty} \left(2\,\frac{e^{(-j^2\,\pi^2\,t)}\,(-1)^j \sin(j\,\pi\,x)}{j\,\pi}\right)\right)$$

The first sum can be simplified using the Fourier sine series for x, yielding

```
> FourierSeries:=Sum(int(2*xi*sin(j*Pi*xi),xi=0..1)
>   *sin(j*Pi*x),j=1..infinity)=x;
```

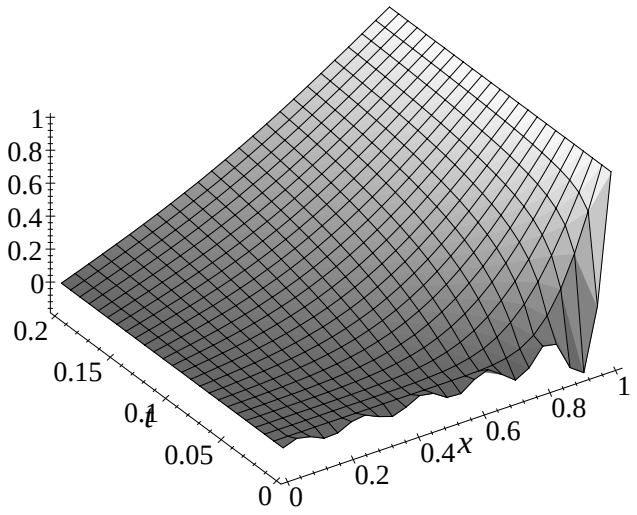
$$FourierSeries := \sum_{j=1}^{\infty} \left(-2\,\frac{(-1)^j \sin(j\,\pi\,x)}{j\,\pi}\right) = x$$

```
> v:=unapply(subs(FourierSeries,""),(x,t));
```

$$v := (x,\ t) \to x + \left(\sum_{j=1}^{\infty} \left(2\,\frac{e^{(-j^2\,\pi^2\,t)}\,(-1)^j \sin(j\,\pi\,x)}{j\,\pi}\right)\right)$$

This is the step response formula that was the starting point in Example 3.2. To plot it, approximate the infinite series with a finite number of terms.

```
> vapprox:=subs(infinity=10,v(x,t)):
> plot3d(vapprox,x=0..1,t=0..1/5,axes=frame,style=
>     patch,shading=greyscale,orientation=[-124,44]);
```



The plot shows the oscillation called *Gibbs's phenomenon* that typically arises when using a truncated eigenfunction expansion of a nonsmooth function. Soon after the initial time, this oscillation is no longer visible, because the term $e^{-j^2\pi^2 t}$ makes the amplitudes of the higher order modes decay much faster than the basic mode.

3.3 Classical Heat Equation

The classical heat equation is

$$\kappa \Delta u - u_t = d(\mathbf{x}, t) \tag{3.6}$$

with u interpreted as temperature and the positive constant κ known as the diffusivity. This is the special case of the diffusion equation with $\mathbf{A} = \kappa \mathbf{I}$ and $c \equiv 0$.

In the Dirichlet boundary condition

$$u(\mathbf{x}, t) = h(\mathbf{x}, t)$$

the input function $h(\mathbf{x}, t)$ is interpreted as an imposed temperature on the boundary. In the Neumann boundary condition

$$\kappa u_{\mathbf{x}} \cdot \mathbf{n} =: \kappa u_n(\mathbf{x}, t) = h(\mathbf{x}, t)$$

$h(\mathbf{x}, t)$ is interpreted as a heat flux, with insulated boundaries modeled by $h \equiv 0$.

The *free field* Green's function is the Green's function for the Dirichlet problem on the unbounded domain $\Omega = \mathsf{R}^n$.

Theorem 3.10 *The free field Green's function for the classical heat equation in n dimensions ($n = 1, 2, 3$) is the symmetric singularity function*

$$F(\mathbf{x}, \mathbf{x}', t) = -\frac{e^{-|\mathbf{x}-\mathbf{x}'|^2/(4\kappa t)}}{(4\pi\kappa t)^{n/2}}$$

PROOF. In one dimension we use the Laplace transform pair [8, p.250]

$$\overbrace{\frac{e^{-\alpha^2/(4t)}}{\sqrt{\pi t}}} = \frac{e^{-\alpha\sqrt{s}}}{\sqrt{s}}$$

to find $\hat{F}$ as

```
> Fhat:=-exp(-abs(x)*sqrt(s/kappa))/2/sqrt(s*kappa);
```

$$Fhat := -\frac{1}{2}\,\frac{e^{(-|x|\sqrt{\frac{s}{\kappa}})}}{\sqrt{s\,\kappa}}$$

where, for convenience, we have moved the origin so that $x' = 0$. We then show that $(\kappa \frac{d^2}{dx^2} - s)\hat{F} = \delta(x)$:

```
> assume(x,real);    interface(showassumed=0):
> simplify(kappa*diff(Fhat,x,x)-s*Fhat);
```

$$\frac{1}{2}\,\frac{\mathrm{signum}(1,\ x)\sqrt{\frac{s}{\kappa}}\,\kappa\,e^{(-|x|\sqrt{\frac{s}{\kappa}})}}{\sqrt{s\,\kappa}}$$

```
> simplify(radsimp(subs(signum(1,x)=2*Dirac(x),")));
```

$$\mathrm{Dirac}(x)$$

In three dimensions we use the Laplace transform pair [8, p.250]

$$\overbrace{\frac{e^{-\alpha^2/(4t)}}{2\sqrt{\pi t^3}}} = \frac{e^{-\alpha\sqrt{s}}}{\alpha}$$

to find $\hat{F}$ in spherical coordinates (with the origin at $\mathbf{x}'$) as

```
> Fhat:=-exp(-r*sqrt(s/kappa))/(4*Pi*kappa*r);
```

$$Fhat := -\frac{1}{4}\,\frac{e^{(-r\sqrt{\frac{s}{\kappa}})}}{\pi\,\kappa\,r}$$

We now verify the three properties of the singularity function of the alternative characterisation given on page 38. First we check that $(\kappa\Delta - s)\hat{F} = 0$ almost everywhere:

```
> simplify(kappa*linalg[laplacian](Fhat,[r,phi,theta],
>       coords=spherical)-s*Fhat);
```

$$0$$

Secondly, we check that $\lim_{\epsilon\to 0}\int_{\partial B_\epsilon}\hat{F}\,\mathrm{d}S = 0$:

```
> limit(int(int(Fhat*r^2*sin(theta),theta=0..Pi),
>     phi=0..2*Pi),r=0);
```

$$0$$

Finally, we check that $\lim_{\epsilon\to 0}\int_{\partial B_\epsilon}(\mathbf{A}\hat{F}_{\mathbf{x}})\cdot\mathbf{n}\,\mathrm{d}S = 1$:

```
> limit(int(int(diff(kappa*Fhat,r)*r^2*sin(theta),
>       theta=0..Pi),phi=0..2*Pi),r=0);
```

$$1$$

In two dimensions we use the Laplace transform pair [3, p.146]

$$\overbrace{t^{-1}e^{-\alpha^2/(4t)}} = 2K_0(\alpha\sqrt{s})$$

to find $\hat{F}$ in polar coordinates (with the origin at $\mathbf{x}'$) as

```
> Fhat:=-BesselK(0,r*sqrt(s/kappa))/(2*Pi*kappa);
```

$$Fhat := -\frac{1}{2}\,\frac{\mathrm{BesselK}(0,\ r\sqrt{\frac{s}{\kappa}})}{\pi\,\kappa}$$

The remaining calculations are similar to the three dimensional case.

```
> simplify(kappa*linalg[laplacian](Fhat,[r,phi],
>       coords=polar)-s*Fhat);

                              0

> limit(int(Fhat*r,phi=0..2*Pi),r=0);

                              0

> limit(int(kappa*diff(Fhat,r)*r,phi=0..2*Pi),r=0);

                              1
```

Finally, for any $t > 0$ and $n = 1, 2, 3$ it is evident that the given singularity functions all tend to zero as $|\mathbf{x} - \mathbf{x}'|$ tends to infinity. ■

Notice that the free field Green's function is strictly positive for all distances and all positive times. Any input applied at a single point at a time $t = t_0$ will affect the solution everywhere at times $t > t_0$, although the effect will be very small at large distance. Nevertheless, the effect of the input travels with infinite speed! This defect in the mathematical model can be corrected by adding an acceleration term to Equation (3.6), yielding a hyperbolic equation of the form

$$\kappa \Delta u - u_t - \gamma u_{tt} = d(\mathbf{x}, t)$$

with $\gamma > 0$. As will be seen in the next chapter, in this PDE local perturbations have finite propagation speed.

Exercises

1. Prove Theorem 3.4.

2. Verify that the problem

$$\begin{aligned} u_{xx} + 2u - tu_t &= 0 \\ u(x,0) &= 0 \quad (0 \le x \le \pi) \\ u(0,t) = u(\pi,t) &= 0 \quad (t > 0) \end{aligned}$$

   has solution $u = \alpha t \sin x$ for all values of the constant α. Why is this not a counterexample to Theorem 3.3?

3. Let $\Omega := (0,\pi) \times (0,\pi)$. Show that the solution of $u_{xx} + u_{yy} - u_t = xy(x-\pi)(y-\pi)$ on $\Omega \times (0,\infty)$, with boundary condition $u = 0$ on $\partial\Omega$ and initial condition $u(x,y,0) = 0$, satisfies the inequality $u(x,y,t) \le \frac{\pi^2}{2}(1-e^{-2t}) \sin x \sin y$ on $\bar{\Omega} \times [0,\infty)$.

4. Prove that the diffusion-convection problem 3.1 with Dirichlet boundary conditions and $c \le 0$ is causal. (*Hint:* examine the uniqueness proofs.)

5. Prove Theorem 3.7.

6. Consider the PDE $u_{xx}(x,t) - u_t(x,t) = 0$ on $x > 0$ and $t > 0$, with zero initial condition and boundary conditions $u(0,t) = h(t)$ and $u(\infty,t) = 0$. Verify that the solution when $h(t) = \mathrm{H}(t)$ is $p(x,t) := \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$. Use Duhamel's Principle to show that the solution for any bounded continuous $h(t)$ is $h * q$ where $q(x,t) := 2xt^{-3/2}e^{-x^2/(4t)}$. This initial-boundary value problem can be interpreted as describing the temperature in the earth at depth x and time t, when the temperature at the surface is h. The functions p and q are the system step and impulse response functions, respectively. Plot them and see how the step or impulse at the surface generates a single wave that seems to propagate into the earth, with the peak value occuring later for points that are deeper. This impression is misleading, however, because the model predicts that the input at the surface travels into the earth infinitely fast. This is seen by noting that the impulse response function is positive at every depth for any positive time.

7. Use the identity

$$\text{Heaviside}(x-y) + \text{Heaviside}(y-x) = 1$$

   to show that the Laplace transform of the Green's function in Exam-

ple 3 can be expressed in the symmetric form

$$\hat{G}(x,y,s) = \begin{cases} -\dfrac{\sinh(1-y)\sqrt{s}\sinh x\sqrt{s}}{\sqrt{s}\sinh\sqrt{s}} & (0 \le x < y \le 1) \\ -\dfrac{\sinh(1-x)\sqrt{s}\sinh y\sqrt{s}}{\sqrt{s}\sinh\sqrt{s}} & (0 \le y < x \le 1) \end{cases}$$

Use the Laplace transform pair [8, p.252]

$$\frac{\cosh x\sqrt{s}}{\sqrt{s}\sinh\sqrt{s}} = \overbrace{1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi^2 t}\cos n\pi x}$$

to find the Green's function in the time domain.

8. Use the results of Exercise 2.18 to find the Green's function for the heat equation

$$\begin{gathered} u_{xx} - u_t = d \quad (0 < x < 1) \\ u(0,t) = h_0(t),\ u'(1,t) = h_1(t) \\ u(x,0) = k(x) \end{gathered}$$

Plot the solution for some particular case of this problem.

9. What values of the constants _C1 and _C2 in the general solution found in the first Maple computations in Example 3 give the singularity function of Theorem 3.10?

10. Find the Green's function for the one dimensional classical heat equation Dirichlet problem on $(0,\infty)\times(0,\infty)$.

Chapter 4

Hyperbolic PDEs

4.1 General Wave Equation

In this chapter we study the initial-boundary value problem

$$\left.\begin{array}{rcll} \mathcal{L}u - e(\mathbf{x},t)u_t - u_{tt} &=& d(\mathbf{x},t) & (\mathbf{x}\in\Omega, t>0) \\ \mathcal{B}u &=& h(\mathbf{x},t) & (\mathbf{x}\in\partial\Omega, t\geq 0) \\ u(\mathbf{x},0) &=& k(\mathbf{x}) & (\mathbf{x}\in\bar{\Omega}) \\ u_t(\mathbf{x},0) &=& l(\mathbf{x}) & (\mathbf{x}\in\bar{\Omega}) \end{array}\right\} \tag{4.1}$$

This problem models linear damped vibration in an nonisotropic medium. Problems with $\Omega = \mathsf{R}^n$ are called *Cauchy problems* and model wave propagation phenomena. We shall refer to problem (4.1) as the wave equation problem.

The wave PDE is a linear second order equation. The PDE's principal part's coefficient matrix is

$$\begin{bmatrix} \mathbf{A} & 0 \\ 0 & -1 \end{bmatrix}$$

which has n positive eigenvalues and 1 negative eigenvalue. The wave PDE is therefore hyperbolic.

We are normally interested in the evolution of the solution of (4.1) as time moves forward from $t = 0$. When $e \equiv 0$ (no damping) the form of the PDE is preserved under a time-reversing coordinate transformation $t' := -t$, so the evolution of the solution backward in time can be studied in the same way as forward evolution.

There is no maximum principle for hyperbolic PDEs. The energy method, however, can still be used to derive well-posedness results.

4.1.1 Well-Posedness Results

We start with the following uniqueness result.

Theorem 4.1 *If the wave equation problem (4.1) is self adjoint with $c \leq 0$, $e \geq 0$, and $\mathcal{L}$ and $\mathcal{B}$ are time invariant, then the solution with Dirichlet, Neumann or Robin boundary conditions (with $fg \geq 0$) is unique.*

PROOF. It suffices to consider the Robin boundary condition; the results for Dirichlet and Neumann conditions follow as special cases. We show that the only solution u of the associated homogeneous problem (in which $d = 0, h = 0, k = 0, l = 0$) is the trivial one. Let $v := u_t$, and apply Green's First Identity (Theorem 2.1) to get

$$\begin{aligned} 0 &= \int_\Omega v(\mathcal{L}u - ev - v_t)\,\mathrm{d}V \\ &= \int_{\partial\Omega} (v\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n}\,\mathrm{d}S + \int_\Omega (-v_{\mathbf{x}}^T \mathbf{A}u_{\mathbf{x}} + cuv - ev^2 - vv_t)\,\mathrm{d}V \end{aligned} \tag{4.2}$$

Partitioning the boundary into disjoint sets $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $g = 0$ on Γ_1 and $g \neq 0$ on Γ_2, the boundary integral in (4.2) can be written

$$\int_{\partial\Omega} (v\mathbf{A}u_{\mathbf{x}}) \cdot \mathbf{n}\,\mathrm{d}S = \int_{\Gamma_2} -\frac{f}{g} uv\,\mathrm{d}S$$

Introducing the energy integral

$$E(t) := \int_{\Gamma_2} \frac{f}{g} u^2\,\mathrm{d}S + \int_\Omega (u_{\mathbf{x}}^T \mathbf{A}u_{\mathbf{x}} - cu^2 + v^2)\,\mathrm{d}V \tag{4.3}$$

equation (4.2) can be written as

$$\dot{E}(t) = -\int_\Omega ev^2\,\mathrm{d}V \tag{4.4}$$

Now $E(0) = 0$, $E(t) \geq 0$ $(t \geq 0)$ and $\dot{E}(t) \leq 0$ $(t \geq 0)$ together imply that $E \equiv 0$. The inequality $E \geq \int_\Omega u_{\mathbf{x}}^T \mathbf{A}u_{\mathbf{x}}\,\mathrm{d}V$ implies that $u_{\mathbf{x}} \equiv 0$, hence u is constant in space, while the inequality $E \geq \int_\Omega v^2\,\mathrm{d}V$ implies that $v \equiv 0$, hence u is constant in time. The zero initial condition then implies $u \equiv 0$. ■

The energy integral $E(t)$ defined by (4.3) can be interpreted as the total energy of the system. The last term $\int_\Omega v^2\,\mathrm{d}V$, represents the kinetic energy, and the remaining terms represent potential energy related to displacements at the boundary and in the domain. If the system has no inputs (that is, $d \equiv 0, h \equiv 0$), then from (4.4) we see that the total energy is nonincreasing in time, so that $E(t) \leq E(0)$. If the system is undamped ($e \equiv 0$), the total energy is constant, and the system is said to be conservative.

Exercise 2.2 gave an example of a pure boundary value problem for a wave PDE that is not well posed, since it does not depend continuously on the boundary data. The discussion in the previous paragraph gives us the following well-posedness result for the initial-boundary value problem.

Theorem 4.2 *The wave equation problem satisfying the conditions of Theorem 4.1 depends continuously on the initial conditions in the sense that* $E(0) \leq \epsilon$ *implies* $E(t) \leq \epsilon$.

Example 1
This is Hadamard's example showing how an *elliptic* initial-boundary value problem may be ill posed. The function $u = n^{-2}\sin(nx)\sinh(nt)$ satisfies the elliptic PDE $u_{xx}+u_{tt} = 0$ and satisfies the boundary conditions $u(0,t) = u(\pi,t) = 0$ and the initial conditions $u(x,0) = 0$, $u_t(x,0) = n^{-1}\sin(nx)$:

```
> u:=(x,t)->sin(n*x)*sinh(n*t)/n^2:
> is(linalg[laplacian](u(x,t),[x,t])=0);
```

true

```
> assume(n,integer);  is(u(0,t)=0) and is(u(Pi,t)=0);
```

true

```
> is(u(x,0)=0) and is(D[2](u)(x,0)=sin(n*x)/n);
```

true

The zero function is a solution of the corresponding homogeneous problem. The following calculations show that, by choosing n sufficiently large, the initial energy of u can be made as small as desired, but for any fixed positive n the energy of the difference between u and the zero function is unbounded:

```
> E:=unapply(simplify(int((D[1](u)(x,t))^2+
>          (D[2](u)(x,t))^2,x=0..1)),t);
```

$$E := t \to \frac{1}{2}\,\frac{-\cos(n)\sin(n) + 2\cosh(n\,t)^2\,n - n}{n^3}$$

```
> limit(E(0),n=infinity);
```

0

```
> assume(n>0);  limit(E(t),t=infinity);
```

∞

4.1.2 Duhamel's Principle

Like the diffusion problem, the wave equation problem is a causal infinite-dimensional linear dynamic system. The state is the function pair (u, u_t), the initial state is (k, l), and the inputs are d and h. The statements and proofs of the versions of Duhamel's Principle given for the diffusion problem (Theorems 3.6 and 3.7) therefore go through almost unchanged for the wave equation problem.

Theorem 4.3 *The solution of the wave equation problem (4.1) with time invariant operators $\mathcal{L}$, $\mathcal{B}$ and time invariant damping coefficient e is given by*

$$u(\mathbf{x}, t) = \frac{\partial}{\partial t} \int_0^t v(\mathbf{x}, t - \tau, \tau)\, \mathrm{d}\tau$$

where, for every fixed $\tau \geq 0$, $v(\mathbf{x}, t, \tau)$ is the solution of the problem with constant inputs

$$\begin{aligned}
\left(\mathcal{L} - e(\mathbf{x})\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2}\right) v(\mathbf{x}, t, \tau) &= d(\mathbf{x}, \tau) \quad (\mathbf{x} \in \Omega, t > 0) \\
\mathcal{B}v(\mathbf{x}, t, \tau) &= h(\mathbf{x}, \tau) \quad (\mathbf{x} \in \partial\Omega, t \geq 0) \\
v(\mathbf{x}, 0, \tau) &= k(\mathbf{x}) \quad (\mathbf{x} \in \bar{\Omega}) \\
v_t(\mathbf{x}, 0, \tau) &= l(\mathbf{x}) \quad (\mathbf{x} \in \bar{\Omega})
\end{aligned}$$

Theorem 4.4 *The solution of the wave equation problem*

$$\begin{aligned}
\left(\mathcal{L} - e(\mathbf{x})\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2}\right) u(\mathbf{x}, t) &= d(\mathbf{x}, t) \quad (\mathbf{x} \in \Omega, t > 0) \\
\mathcal{B}u(\mathbf{x}, t) &= 0 \quad (\mathbf{x} \in \partial\Omega, t > 0) \\
u(\mathbf{x}, 0) &= 0 \quad (\mathbf{x} \in \Omega) \\
u_t(\mathbf{x}, 0) &= 0 \quad (\mathbf{x} \in \Omega)
\end{aligned}$$

with time invariant operators $\mathcal{L}$, $\mathcal{B}$ is given by

$$u(\mathbf{x}, t) = \int_0^t v(\mathbf{x}, t - \tau, \tau)\, \mathrm{d}\tau$$

where, for every fixed $\tau \geq 0$, $v(\mathbf{x}, t, \tau)$ is the solution of the problem

$$\begin{aligned}
\left(\mathcal{L} - e(\mathbf{x})\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2}\right) v(\mathbf{x}, t, \tau) &= 0 \quad (\mathbf{x} \in \Omega, t > 0) \\
\mathcal{B}v(\mathbf{x}, t, \tau) &= 0 \quad (\mathbf{x} \in \partial\Omega, t > 0) \\
v(\mathbf{x}, 0, \tau) &= 0 \quad (\mathbf{x} \in \Omega) \\
v_t(\mathbf{x}, 0, \tau) &= -d(\mathbf{x}, \tau) \quad (\mathbf{x} \in \Omega)
\end{aligned}$$

4.1.3 Green's Functions

For convenience we continue to consider the wave equation problem with time invariant damping coefficient e and operators $\mathcal{L}$, $\mathcal{B}$ time invariant. As we did for the diffusion problem, we take the Laplace transform of the wave equation, obtaining

$$\begin{aligned}
\left(\mathcal{L} - e(\mathbf{x})s - s^2\right)\hat{u}(\mathbf{x}, s) &= \hat{d}(\mathbf{x}, s) - sk(\mathbf{x}) - l(\mathbf{x}) - e(\mathbf{x})k(\mathbf{x}) \quad (\mathbf{x} \in \Omega) \\
\mathcal{B}\hat{u}(\mathbf{x}, s) &= \hat{h}(\mathbf{x}, s) \quad (\mathbf{x} \in \partial\Omega)
\end{aligned}$$

The initial conditions are now symbolically included in the PDE, and the problem has the form of a boundary value problem. We then find the singularity function as the inverse Laplace transform of a solution of the PDE

$$\left(\mathcal{L} - es - s^2\right)\hat{F}(\mathbf{x}, \mathbf{x}', s) = \delta(\mathbf{x} - \mathbf{x}')$$

and take the Green's function to be the inverse Laplace transform of $\hat{G} := \hat{F} + \hat{H}$ where $\hat{H}$ is the solution to the boundary value problem

$$\left(\mathcal{L} - e(\mathbf{x})s - s^2\right)\hat{H} = 0 \text{ in } \Omega,\ \mathcal{B}\hat{H} = -\mathcal{B}\hat{F} \text{ on } \partial\Omega$$

Example 2
Consider the one dimensional wave equation

$$u_{xx} - u_{tt} = d(x,t)$$

on $(0,1) \times (0,\infty)$ with Dirichlet boundary conditions. This is a model of a vibrating string with prescribed end displacements. First we find a singularity function.

```
> U:=expand(rhs(dsolve(diff(u(x),x,x)-s^2*u(x)=
>     Dirac(x-y),u(x))));
```

$$U := \frac{1}{2}\frac{\text{Heaviside}(x-y)\,e^{(s\,x)}}{s\,e^{(s\,y)}} - \frac{1}{2}\frac{\text{Heaviside}(x-y)\,e^{(s\,y)}}{s\,e^{(s\,x)}} + \_C1\,e^{(s\,x)} + \frac{\_C2}{e^{(s\,x)}}$$

The following choice of the constants gives a symmetric singularity function.

```
> subs(_C1=-(Heaviside(x-y)+Heaviside(y-x))/exp(y*s)
>       /s/2,_C2=0,U);
```

$$\frac{1}{2}\frac{\text{Heaviside}(x-y)\,e^{(s\,x)}}{s\,e^{(s\,y)}} - \frac{1}{2}\frac{\text{Heaviside}(x-y)\,e^{(s\,y)}}{s\,e^{(s\,x)}} - \frac{1}{2}\frac{(\text{Heaviside}(x-y)+\text{Heaviside}(y-x))\,e^{(s\,x)}}{e^{(s\,y)}\,s}$$

```
> Fhat:=unapply(collect(",[exp,Heaviside]),(x,y));
```

$$Fhat := (x,\ y) \to -\frac{1}{2}\frac{\text{Heaviside}(y-x)\,e^{(s\,x)}}{s\,e^{(s\,y)}} - \frac{1}{2}\frac{\text{Heaviside}(x-y)\,e^{(s\,y)}}{s\,e^{(s\,x)}}$$

Check that it is a singularity function for the wave equation and that it is symmetric.

```
> simplify(diff(Fhat(x,y),x,x)-s^2*Fhat(x,y));
```

$$\text{Dirac}(x-y)$$

```
> is(Fhat(x,y)=Fhat(y,x));
```

true

Next, find the regular part of the Green's function for Dirichlet boundary conditions and check that it is symmetric.

```
> assume(0<Y,Y<1);
> dsolve({diff(u(x),x,x)-s^2*u(x)=0,u(0)=-Fhat(0,Y),
>    u(1)=-Fhat(1,Y)},u(x));
```

$$u(x) = \frac{1}{2}\,\frac{((e^{(s\,Y)})^2-1)\,e^{(s\,x)}}{s\,e^{(s\,Y)}\,(-1+(e^{s})^2)} - \frac{1}{2}\,\frac{(-(e^{s})^2+(e^{(s\,Y)})^2)\,e^{(-s\,x)}}{s\,e^{(s\,Y)}\,(-1+(e^{s})^2)}$$

```
> Hhat:=unapply(simplify(subs(Y=y,rhs(")))),(x,y));
```

$$Hhat := (x,\ y) \to -\frac{1}{2}\,\frac{-e^{(s\,(x+y))}+e^{(s\,(x-y))}-e^{(-s\,(x+y-2))}+e^{(-s\,(x-y))}}{s\,(-1+e^{(2\,s)})}$$

```
> expand(Hhat(x,y)-Hhat(y,x));
```

$$0$$

From tables we find the Laplace transform pairs

$$\frac{e^{-\beta s}}{s} = \overbrace{\mathrm{H}(t-\beta)} \quad (\beta \geq 0)$$

and

$$\frac{e^{-\beta s}}{s(1-e^{-\alpha s})} = \overbrace{\sum_{n=0}^{\infty} \mathrm{H}(t-\alpha n-\beta)} \quad (\alpha > 0, \beta \geq 0)$$

(see Figure 4.1) so we conclude that

$$\begin{aligned} G(x,y,t) \;=\; & -\frac{1}{2}\mathrm{H}(t-|x-y|) \\ & -\frac{1}{2}\sum_{n=0}^{\infty}[-\mathrm{H}(t-2n-(2-x-y)) \\ & +\mathrm{H}(t-2n-(2-x+y))-\mathrm{H}(t-2n-(x+y)) \\ & +\mathrm{H}(t-2n-(2+x-y))] \end{aligned}$$

Note that for any fixed t, the infinite series appearing in the above formula reduces to a finite sum, since taking n sufficiently large makes the arguments of the Heaviside functions negative.

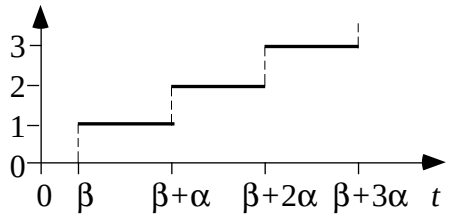


Figure 4.1: The inverse Laplace transform of $s^{-1}e^{-\beta s}(1-e^{-\alpha s})^{-1}$.

Applying Theorem 2.15, we find the solution of the self-adjoint wave problem with Robin boundary condition in the form

$$\begin{aligned}\hat{u}(\mathbf{x}', s) &= \int_\Omega \hat{G}(\mathbf{x}, \mathbf{x}', s)(\hat{d}(\mathbf{x}, s) - sk(\mathbf{x}) - l(\mathbf{x}) - e(\mathbf{x})k(\mathbf{x}))\,\mathrm{d}V \\ &\quad + \int_{\Gamma_1} \frac{\hat{h}(\mathbf{x}, s))}{f(\mathbf{x})} \left(\mathbf{A}(\mathbf{x})\hat{G}_{\mathbf{x}}(\mathbf{x}, \mathbf{x}', s)\right) \cdot \mathbf{n}\,\mathrm{d}S \\ &\quad - \int_{\Gamma_2} \hat{G}(\mathbf{x}, \mathbf{x}', s)\frac{\hat{h}(\mathbf{x}, s)}{g(\mathbf{x})}\,\mathrm{d}S\end{aligned}$$

Taking the inverse Laplace transform of both sides gives

Theorem 4.5 *The solution of the time invariant self-adjoint wave equation problem with Robin boundary conditions is*

$$\begin{aligned}u(\mathbf{x}', t') &= \int_0^{t'} \int_\Omega G(\mathbf{x}, \mathbf{x}', t' - t)d(\mathbf{x}, t)\,\mathrm{d}V\,\mathrm{d}t \\ &\quad - \frac{\partial}{\partial t'} \int_\Omega G(\mathbf{x}, \mathbf{x}', t')k(\mathbf{x})\,\mathrm{d}V \\ &\quad - \int_\Omega G(\mathbf{x}, \mathbf{x}', t')[l(\mathbf{x}) + e(\mathbf{x})k(\mathbf{x})]\,\mathrm{d}V \\ &\quad + \int_0^{t'} \int_{\Gamma_1} \frac{h(\mathbf{x}, t))}{f(\mathbf{x})} (\mathbf{A}(\mathbf{x})G_{\mathbf{x}}(\mathbf{x}, \mathbf{x}', t' - t)) \cdot \mathbf{n}\,\mathrm{d}S\,\mathrm{d}t \\ &\quad - \int_0^{t'} \int_{\Gamma_2} G(\mathbf{x}, \mathbf{x}', t' - t)\frac{h(\mathbf{x}, t)}{g(\mathbf{x})}\,\mathrm{d}S\,\mathrm{d}t\end{aligned}$$

For a Dirichlet problem the last two terms reduce to

$$\int_0^{t'} \int_{\partial\Omega} h(\mathbf{x}, t)\,(\mathbf{A}G_{\mathbf{x}}(\mathbf{x}, \mathbf{x}', t' - t)) \cdot \mathbf{n}\,\mathrm{d}S\,\mathrm{d}t$$

while for a Neumann problem the last two terms of the Robin problem solution reduce to

$$-\int_0^{t'} \int_{\partial\Omega} G(\mathbf{x}, \mathbf{x}', t' - t)h(\mathbf{x}, t)\,\mathrm{d}S\,\mathrm{d}t$$

Example 2 (continued)
The Green's function for this one dimensional wave equation Cauchy problem is just the singularity function

$$G(x, x', t) = F(x, x', t) = -\frac{1}{2}\mathrm{H}(t - |x - x'|)$$

Applying the formula from Theorem 4.5 gives

$$u(x', t') = -\frac{1}{2}\int_0^{t'} \int_{-\infty}^{\infty} \mathrm{H}(t' - t - |x - x'|)d(x, t)\,\mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{-\infty}^{\infty}[\delta(t'-|x-x'|)k(x)+\mathrm{H}(t'-|x-x'|)l(x)]\,\mathrm{d}x \\
=\ &-\frac{1}{2}\int_0^{t'}\int_{x'-(t'-t)}^{x'+(t'-t)} d(x,t)\,\mathrm{d}x\mathrm{d}t+\frac{1}{2}[k(x'+t')+k(x'-t')] \\
&+\frac{1}{2}\int_{x'-t'}^{x'+t'} l(x)\,\mathrm{d}x
\end{aligned}
$$

This formula is known as *d'Alembert's solution*.

4.1.4 Method of Eigenfunction Expansion

As for the diffusion problem, the Green's function for the wave equation can be expressed as a series. For simplicity we only give the formula for the undamped equation.

Theorem 4.6 *The Green's function for the time invariant self-adjoint wave equation problem (4.1) is given by*

$$G(\mathbf{x},\mathbf{x}',t)=-\sum_{j=1}^{\infty}\frac{\phi_j(\mathbf{x})\phi_j(\mathbf{x}')}{\|\phi_j\|^2\sqrt{\lambda_j}}\sin\sqrt{\lambda_j}t$$

where $\{(\lambda_j,\phi_j):j=1\ldots\infty\}$ *is the set of eigenvalues and eigenfunctions from*

$$\mathcal{L}\phi+\lambda\phi=0,\quad \mathcal{B}\phi=0$$

The proof is similar to that of Theorem 3.9 (Exercise 4).

Because of the term $\sin\sqrt{\lambda_j}t$ appearing in the Green's function, the values $\sqrt{\lambda_j}/(2\pi)$ are called *natural frequencies* in vibration problems. The eigenfunctions are called *mode shapes*.

Example 3
Consider the 1D wave equation with Dirichlet boundary condition. From example 2.5 we have the eigenvalues and eigenvectors

```
> lambda:=j->j^2*Pi^2;
```

$$\lambda := j \to j^2\pi^2$$

```
> mu:=unapply(radsimp(sqrt(lambda(j))),j);
```

$$\mu := j \to \pi\, j$$

```
> phi:=unapply(sin(mu(j)*x),j,x);
```

$$\phi := (j,\ x) \to \sin(\pi\ j\ x)$$

The Green's function is therefore

```
> assume(j,integer): interface(showassumed=0):
> Gterm:=unapply(-phi(j,x)*phi(j,y)*sin(mu(j)*t)
> /mu(j)/int(phi(j,x)^2,x=0..1),(x,y,t)):
> G:=Sum(Gterm(x,y,t),j=1..infinity);
```

$$G := \left(\sum_{j=1}^{\infty} (-2 \frac{\sin(\pi\ j\ x) \sin(\pi\ j\ y) \sin(\pi\ j\ t)}{\pi\ j}) \right)$$

If the initial conditions are zero and the input is a unit impulse $h_0 = \delta(t)$ at $x = 0$ and $t = 0$, then the solution is

```
> assume(t>0):
> u:=Sum(int(Dirac(tau)*D[1](Gterm)(0,y,t-tau),tau=0..t),
>      j=1..infinity);
```

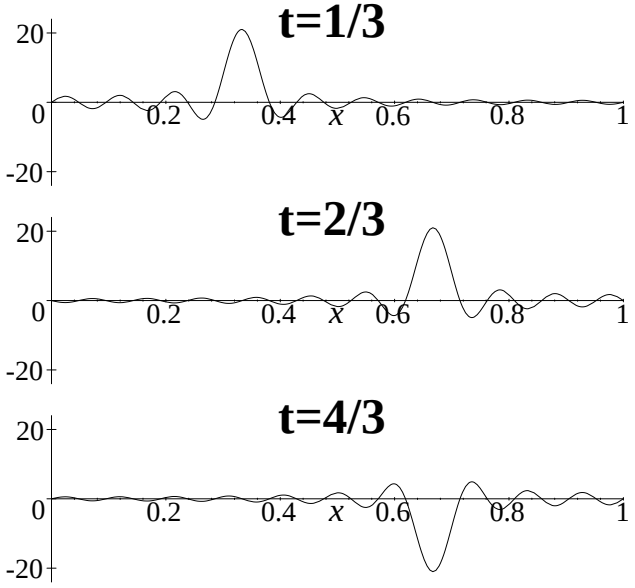
$$u := \sum_{j=1}^{\infty} (2 \sin(\pi\ j\ y) \sin(\pi\ j\ t))$$

Finally we plot the 20-mode approximation at various times.

```
> uApprox:=unapply(subs(infinity=N,u),(N,y,t));
```

$$uApprox := (N,\ y,\ t) \rightarrow \sum_{j=1}^{N} (2 \sin(\pi\ j\ y) \sin(\pi\ j\ t))$$

```
> N:=20:
> for t in [1/3,2/3,4/3] do
>  plot(uApprox(N,x,t),x=0..1,-1.2*N..1.2*N,title=
>  cat('t=',convert(t,string)),ytickmarks=[-N,0,N])
> od;
```



Here we can see how the pulse moves with unit speed to the right until it comes to the boundary. There it is reflected and starts to come back with its shape reversed. The Gibbs phenomenon is clearly visible.

4.2 The Classical Wave Equation

The classical wave equation is the PDE

$$\Delta u - u_{tt} = d \tag{4.5}$$

It is used to describe undamped linear wave propagation and vibration phenomena in isotropic media. The Green's function for wave propagation problems is given by

Theorem 4.7 *The Green's function for the classical wave equation Cauchy problem is the singularity function*

$$F(x, x', t) = -\frac{1}{2}\mathrm{H}(t - |x - x'|)$$

in one dimension,

$$F(\mathbf{x}, \mathbf{x}', t) = -\frac{1}{2\pi\sqrt{t^2 - |\mathbf{x} - \mathbf{x}'|^2}}\mathrm{H}(t - |\mathbf{x} - \mathbf{x}'|)$$

in two dimensions, and

$$F(\mathbf{x}, \mathbf{x}', t) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}\delta(t - |\mathbf{x} - \mathbf{x}'|)$$

in three dimensions

PROOF. The formula for the one dimensional wave equation was derived in Example 2. In three dimensions we use the Laplace transform pair

$$\overbrace{\delta(t - \alpha)} = e^{-\alpha s}$$

to find $\hat{F}$ in spherical coordinates (with the origin at $\mathbf{x}'$) as

```
> Fhat:=-exp(-r*s)/(4*Pi*r);
```

$$Fhat := -\frac{1}{4}\,\frac{e^{(-r\,s)}}{r\,\pi}$$

We now verify the three properties of the singularity function of the alternative characterisation given on page 38. First we check that $(\Delta - s^2)\hat{F} = 0$ almost everywhere:

```
> simplify(linalg[laplacian](Fhat,[r,phi,theta],
>       coords=spherical)-s^2*Fhat);
```

$$0$$

Secondly, we check that $\lim_{\epsilon\to 0}\int_{\partial B_\epsilon}\hat{F}\,\mathrm{d}S = 0$:

```
> limit(int(int(Fhat*r^2*sin(theta),theta=0..Pi),
>     phi=0..2*Pi),r=0);

                              0
```

Finally, we check that $\lim_{\epsilon\to 0}\int_{\partial B_\epsilon}(\hat{F}_{\mathbf{x}})\cdot\mathbf{n}\,\mathrm{d}S = 1$:

```
> limit(int(int(diff(Fhat,r)*r^2*sin(theta),
>       theta=0..Pi),phi=0..2*Pi),r=0);

                              1
```

We now use the singularity function for the three dimensional problem to derive the singularity function for the two dimensional problem. This projection technique is known as the *method of descent*.

Substituting the singularity function into the solution formula (Theorem 4.5) for a three dimensional wave equation with $k \equiv 0$ and $d \equiv 0$ gives *Kirchhoff's formula*

$$\begin{aligned} u(\mathbf{0},t) &= \frac{1}{4\pi}\int_\Omega \delta(t-|\mathbf{x}|)l(\mathbf{x})\,\mathrm{d}V \\ &= \frac{1}{4\pi t}\int_{\partial B_t} l(\mathbf{x})\,\mathrm{d}S \\ &= \frac{t}{4\pi}\int_0^\pi\int_0^{2\pi} l(t,\theta,\phi)\sin\theta\,\mathrm{d}\phi\mathrm{d}\theta \end{aligned}$$

Here the integral is taken over the surface of a sphere of radius t centered at the origin.

For a two dimensional problem, l is invariant with respect to the third spatial dimension x_3, and Kirchhoff's formula can be transformed (taking $r = t\sin\theta$) to

$$\begin{aligned} u(\mathbf{0},t) &= \frac{1}{2\pi}\int_0^t\int_0^{2\pi}\frac{r\,l(r,\phi)}{\sqrt{t^2-r^2}}\,\mathrm{d}\phi\mathrm{d}r \\ &= \frac{1}{2\pi}\iint_{r\le t}\frac{l(\mathbf{x})}{\sqrt{t^2-x_1^2-x_2^2}}\,\mathrm{d}x_1\mathrm{d}x_2 \end{aligned}$$

which is the formula corresponding to the singularity function for the two dimensional wave problem. ■

The first thing to notice about the wave equation's singularity function is that it is zero outside a ball (interval in 1D, disk in 2D) whose radius is t. This means that initial values at a point $\mathbf{x}_0$ will have no influence on the solution at points outside the ball of radius t centered at that point. This expanding ball defines a cone in $\mathsf{R}^n \times [0,\infty)$ known as the *range of influence* of the initial point $\mathbf{x}_0$. In contrast to the diffusion equation, where disturbances have an infinite speed of propagation, disturbances in a wave equation have a finite speed of propagation.

Looking at the same concept from a different point of view, we observe that the solution at a point $\mathbf{x}_1$ and time $T \geq 0$ will not be influenced by initial values outside the ball of radius T centered at that point, nor by an input $d(\mathbf{x}, t)$ with $\mathbf{x}$ outside the ball of radius $T - t$. This shrinking ball defines a cone known as the *domain of dependence* of the solution point $u(\mathbf{x}_1, T)$. The cones for 2D wave problems are shown in Figure 4.2.

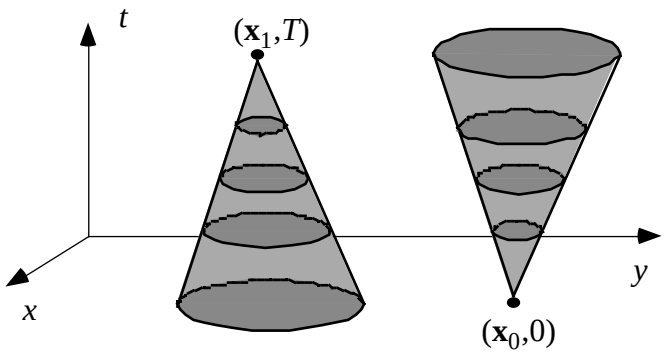


Figure 4.2: Domain of dependence of $u(\mathbf{x}_1, T)$ and range of influence of $\mathbf{x}_0$.

Example 4

Let's look at the function $-F(x, 0, t)$, which is the solution of the classical wave equation in an unbounded domain, with zero initial displacement, zero initial velocity, and input $d(x, t) = \delta(x)\delta(t)$.

In three dimensions the solution in spherical coordinates is $\delta(t-r)/(4\pi r)$. This is a singularity in the shape of the surface of an expanding sphere. The radius of the sphere is growing at unit speed.

A distinguishing feature of this Green's function is that the wave front remains infinitesimally thin. An observer at a fixed point in space would, after a delay, detect an impulse; after that, the solution returns to zero. The moving wave front leaves no trace behind it. One consequence of this is that a disturbance of finite time duration from a point source is observed as a signal of exactly the same duration. This property of the wave equation in three dimensions is known as *Huygens's principle*.

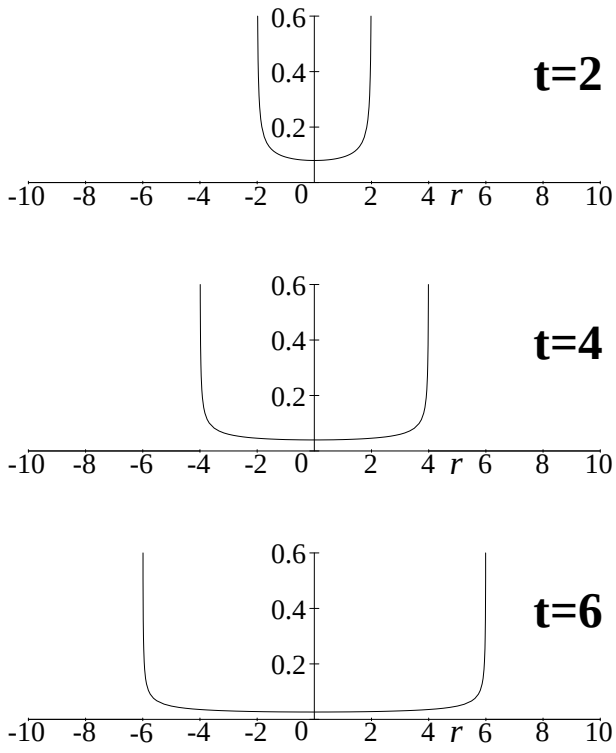
In two dimensions the solution in polar coordinates is

```
> u:=(r,t)->Heaviside(t-abs(r))/sqrt(t^2-r^2)/2/Pi;
```

$$u := (r,\ t) \rightarrow \frac{1}{2}\,\frac{\text{Heaviside}(t - |r|)}{\sqrt{t^2 - r^2}\,\pi}$$

Here the solution is nonzero on a disk whose radius grows with unit speed. There is a singularity on the disk boundary. Let's plot the solution as a function of radius, at various times.

```
> with(plots):
> waveplot:=t->plot(u(r,t),r=-10..10,thickness=4,
>   view=[0..10,0..0.6],discont=true):
> wavetext:=t->textplot([9,0.4,cat('t=',convert(t,
>  string))],font=[TIMES,BOLD,18]):
> for t in [2,4,6] do
>    display({waveplot(t),wavetext(t)})
> od;
```



Here we see how, after the front has passed, the solution is not zero. Huygens's principle does not hold in two dimensions.

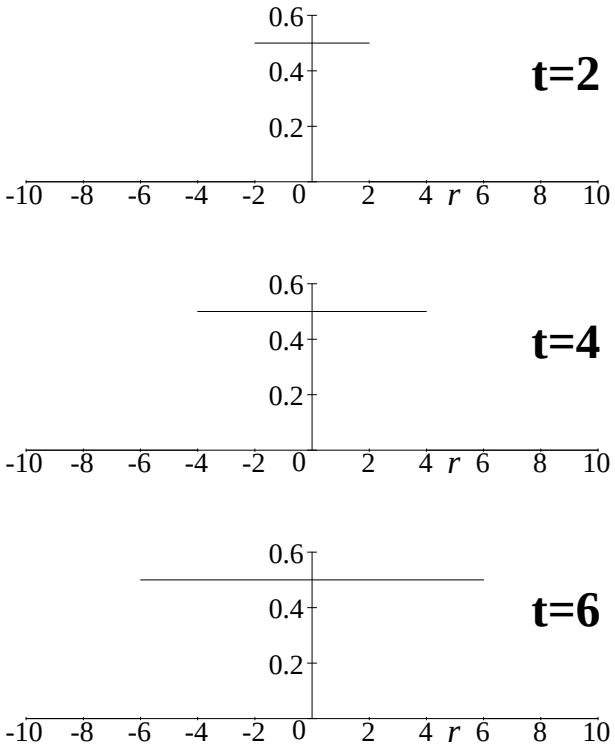
Using the method of descent, the two dimensional solution can be thought of as a special three dimensional problem that is invariant with respect to z. The input is then a line of impulses along the z-axis, and the wave front is the surface of a cylinder. After an initial delay, an observer at a fixed point would detect the singularity caused by the impulse that originated at the nearest z-axis point. After that, the impulses from further up and down the z axis are detected, with decreasing amplitude because of increasing distance.

In one dimension we have

```
> u:=(x,t)->Heaviside(t-abs(x))/2;
```

$$u := (x,\ t) \to \frac{1}{2}\,\text{Heaviside}(t - |x|)$$

```
> for t in [2,4,6] do
>    display({waveplot(t),wavetext(t)})
> od;
```



We see how the solution is a jump discontinuity that propagates with unit speed. The solution remains constant after the jump has passed. Again, Huygens's principle does not hold.

The interpretation of the solution using the method of descent is that in three dimensions the input is a plane of impulses. The observer detects a jump when the impulse from the nearest point on the plane arrives. The impulses from other points of the plane arrive later. The solution remains constant, as attenuation effect due to greater distance is exactly balanced by the larger number of impulses that arrive.

Exercises

1. Prove that if $v(x,t)$ is the solution of the Cauchy problem

$$\begin{aligned} \mathcal{L}v - e(\mathbf{x})v_t - v_{tt} &= e(\mathbf{x})l(\mathbf{x})\mathrm{H}(t) \\ v(\mathbf{x},0) &= l(\mathbf{x}) \\ v_t(\mathbf{x},0) &= 0 \end{aligned}$$

with time invariant $\mathcal{L}$, $\mathcal{B}$, and e, then $u := \int_0^t v(t-t')\,\mathrm{d}t'$ is the solution of

$$\begin{aligned} \mathcal{L}u - e(\mathbf{x})u_t - u_{tt} &= 0 \\ u(\mathbf{x},0) &= 0 \\ u_t(\mathbf{x},0) &= l(\mathbf{x}) \end{aligned}$$

2. Find and plot the solution of the one dimensional wave equation Cauchy problem $c^2u_{xx} - u_{tt} = 0$ with initial conditions $u(x,0) = \alpha\sin(\omega x)$, $u_t(x,0) = 0$. Interpret the solution as a standing wave and find its amplitude, frequency, and nodes. Repeat for the Cauchy problem $c^2u_{xx} - u_{tt} = 0$ with initial conditions $u(x,0) = 0$, $u_t(x,0) = \alpha\sin(\omega x)$ and for the Cauchy problem $c^2u_{xx} - u_{tt} = -\alpha\sin(\omega x)$ with initial conditions $u(x,0) = 0$, $u_t(x,0) = 0$. Here c is a positive constant (the speed of propagation); a rescaling of the time variable reduces the wave equation to the form studied in Example 2.

3. Use the two formulas for the Green's function from examples 2 and 3 to find the value of $u(\frac{1}{4}, \frac{3}{4})$ when the function $u(x,t)$ is the solution of the vibrating string problem

$$\begin{aligned} u_{xx} - u_{tt} = 2 \quad & (0 < x < 1, t > 0) \\ u(x,0) = u_t(x,0) = 0 \quad & (0 \le x \le 1) \\ u(0,t) = u(1,t) = t \quad & (t \ge 0) \end{aligned}$$

4. Prove Theorem 4.6.

5. For the wave equation problem $u_{xx} - u_{tt} = 0$ on $(0,\infty)\times(0,\infty)$ with boundary condition $u(0,t) = 0$ and initial conditions $u(x,0) = k(x)$, $u_t(x,0) = l(x)$, derive a solution formula analogous to d'Alembert's solution.

6. Find the solution of the classical wave equation in R^n for $n = 1, 2, 3$ with zero initial conditions and input $d(\mathbf{x},t) = -\delta(\mathbf{x})\sin(\omega t)\mathrm{H}(t)$. Plot its value as a function of time at various fixed observation points. Show that in two dimensions, the solution tends to a standing wave of the form

$$u(r,\infty) = \frac{1}{2\pi}\int_1^\infty \frac{\sin(\omega\theta r)}{\sqrt{\theta^2-1}}\,\mathrm{d}\theta = \frac{1}{4}J_0(\omega r)$$

According to this result, a stationary observer eventually does not detect any oscillation!

7. Using the Laplace transform pair [3, p.250]

$$\frac{e^{-\tau\sqrt{s(s+\alpha)}}}{\sqrt{s(s+\alpha)}} = \overbrace{\mathrm{H}(t-\tau)e^{-\alpha t/2}I_0(\alpha\sqrt{t^2-\tau^2})} \quad (\tau \geq 0, \alpha > 0)$$

find and plot the Cauchy problem's Green's function for the one dimensional classical wave equation with a constant damping coefficient e. Does the addition of damping affect the range of influence?

8. Use d'Alembert's formula to solve the one dimensional classical wave equation Cauchy problem with no input ($d \equiv 0$), zero initial velocity ($l \equiv 0$), and initial shape $k(x) = \mathrm{H}(1-2|x|)\cos(\pi x)$. Plot snapshots of the solution at various fixed times.

9. Let F be the Green's function for a wave equation in an unbounded domain. A boundary operator $\mathcal{B}$ for a finite domain Ω that satisfies $\mathcal{B}F \equiv 0$ is called an *absorbing boundary* for this wave equation, since the solution inside the domain is identical to the solution for an unbounded domain. In transmission line theory this is called a *matched impedance*. There is no reflection of waves from this boundary: impinging waves are completely absorbed. Show that the absorbing boundary condition for a one dimensional classical wave equation on $0 < x < 1$ is given by the Robin boundary conditions $u_t(0,t) - u_x(0,t) = 0$ and $u_t(1,t) + u_x(1,t) = 0$.

Chapter 5

First Order PDEs

5.1 Single Quasilinear First Order PDE in Two Variables

5.1.1 Characteristic Curves

A quasilinear first order PDE for a function $u(x, y)$ of two variables has the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{5.1}$$

If $c \equiv 0$ the PDE is *homogeneous*. If a and b do not depend on u then it is *almost linear*. If additionally c is linear in u then the PDE is *linear*.

Let $\mathbf{r} := [x, y, z]$ denote position in R^3. The vector field $\mathbf{a}(\mathbf{r})$ given by $\mathbf{a} := [a(\mathbf{r}), b(\mathbf{r}), c(\mathbf{r})]$ is called the *direction field* of the PDE. We assume that $\mathbf{a}$ is continuously differentiable in some domain. Also, to ensure that the PDE does not degenerate to an algebraic equation, it is assumed that $a(\mathbf{r})$ and $b(\mathbf{r})$ are not simultaneously zero anywhere in this domain.

Geometrically, a solution of the PDE (5.1) is a surface

$$z = u(x, y) \tag{5.2}$$

called an *integral surface* of the PDE. It is a level surface of the scalar field $F(\mathbf{r}) := u(x, y) - z$. A normal to the integral surface is given by $\nabla F = [u_x, u_y, -1]$. The PDE (5.1) rewritten in the form

$$\mathbf{a} \cdot [u_x, u_y, -1] = 0 \tag{5.3}$$

can then be interpreted as saying that the normal of an integral surface is orthogonal to the direction field $\mathbf{a}$. In other words, an integral surface is tangential to the direction field.

The *field lines* of a vector field are curves that are everywhere tangential to the vector field. A curve $\Gamma : \mathbf{r} = \mathbf{R}(t)$ is a field line of the PDE direction field $\mathbf{a}$ if it satisfies a vector ODE of the form

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = k(t)\mathbf{a}(\mathbf{R}) \tag{5.4}$$

where $k(t)$ is a continuous function that is nowhere zero. Field lines of the PDE direction field are called *characteristic curves.*

From ODE theory we know that when $\mathbf{a}$ is continuously differentiable then for every point $\mathbf{r}_0$ of the domain there is a unique curve satisfying vector ODE (5.4) and containing $\mathbf{r}_0$. The family of solutions for (5.4) has three free constants, corresponding to the three component ODEs. However, one of these constants can be related to the curve parametrisation without changing the curves, as follows. Consider the new curve parametrisation $s = \sigma(t)$ with $\sigma' = k$. Denoting $\bar{\mathbf{R}} \circ \sigma := \mathbf{R}$ and $\bar{\mathbf{a}} \circ \bar{\mathbf{R}} := \mathbf{a}$, the ODE (5.4) is transformed into

$$\frac{\mathrm{d}\bar{\mathbf{R}}}{\mathrm{d}s} = \bar{\mathbf{a}}(\bar{\mathbf{R}}) \tag{5.5}$$

This ODE is autonomous, that is, the parameter s does not appear explicitly in the right hand side. Consequently, $\bar{\mathbf{R}}(s)$ and $\bar{\mathbf{R}}(s-C)$ give the same curve for any constant C. This C corresponds to one of the three free constants of the family of solutions of the autonomous ODE (5.5). The curves themselves, whose shape is independent of the parametrisation, are thus specified by the two remaining constants.

Another way of seeing why the characteristic curves make up a two parameter family is to write the ODE system (5.4) in the form

$$\frac{\mathrm{d}x}{a(x,y,z)} = \frac{\mathrm{d}y}{b(x,y,z)} = \frac{\mathrm{d}z}{c(x,y,z)} \tag{5.6}$$

This is an abuse of notation, since some of the denominators may be zero. However, this formal defect can always be corrected: if $a \neq 0$, (5.6) can be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x,y,z)}{a(x,y,z)}, \quad \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{c(x,y,z)}{a(x,y,z)}$$

and similarly if $b \neq 0$. However, (5.6) is a customary and convenient way of writing the system. Since there are now two ODEs, the solution set is a two parameter family of curves.

In the case of an almost linear first order PDE

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u)$$

the characteristic ODEs (5.6) are

$$\frac{\mathrm{d}x}{a(x,y)} = \frac{\mathrm{d}y}{b(x,y)} = \frac{\mathrm{d}z}{c(x,y,z)}$$

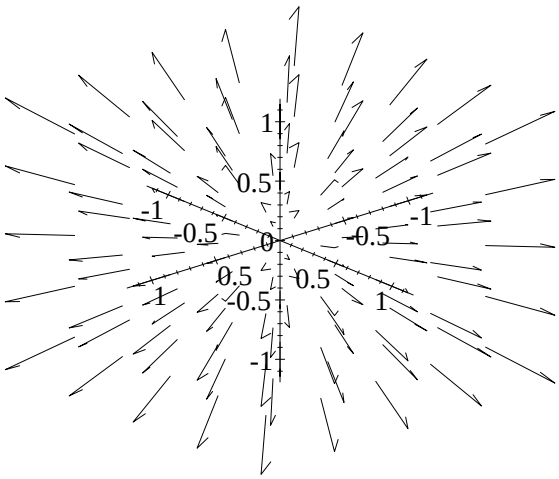
The first of these equations can be solved for the projections of the characteristic curves onto the xy plane, which are called *base characteristic curves* of the PDE. The z component of the characteristic curves is then found by solving the remaining ODE.

Example 1
For the linear first order PDE

$$x\,u_x + y\,u_y = u$$

the direction field is $[x,\ y,\ z]$. These vectors are all pointing away from the origin.

```
> with(plots):
> a:=(x,y,z)->[x,y,z]:
> fieldplot3d(a(x,y,z),x=-1..1,y=-1..1,z=-1..1,
> grid=[5,5,5],axes=normal,orientation=[49,63]);
```



If we choose $k(t) = 1/t$ then the three ODEs that define the characteristic curves are

```
> k:=t->1/t:
> r:=[x,y,z]:
> for i from 1 to 3 do
>  ODE[i]:=diff(r[i](t),t)=k(t)*a(x(t),y(t),z(t))[i]
> od;
```

$$ODE_1 := \frac{\partial}{\partial t}\,\mathrm{x}(t) = \frac{\mathrm{x}(t)}{t}$$

$$ODE_2 := \frac{\partial}{\partial t}\,\mathrm{y}(t) = \frac{\mathrm{y}(t)}{t}$$

$$ODE_3 := \frac{\partial}{\partial t}\,\mathrm{z}(t) = \frac{\mathrm{z}(t)}{t}$$

The general ODE solution is

```
> gensol:=dsolve({ODES},{x(t),y(t),z(t)},explicit);
```

$$gensol := \{\mathrm{y}(t) = t\ \_C2,\ \mathrm{x}(t) = t\ \_C1,\ \mathrm{z}(t) = t\ \_C3\}$$

Letting $C_1 = 1$ gives the following family of characteristic curves.

```
> Characteristics:=subs(gensol,_C1=1,[x(t),y(t),z(t)]);
```

$$Characteristics := [t,\ t\_C2,\ t\_C3]$$

These characteristic curves are parametrised by t; different curves correspond to different values of the two constants C_2 and C_3.

If a surface of the form $z = u(x, y)$ is a union of characteristic curves, then it is an integral surface, since it is tangent to the PDE direction field. The converse also holds: any integral surface is the union of characteristic curves. This follows from the following

Theorem 5.1 *Through every point of an integral surface there passes a characteristic curve contained in the surface.*

PROOF. Let $\Sigma : z = u(x, y)$ be an integral surface, let $\mathbf{r}_0$ be a point on Σ, and let $\Gamma : \mathbf{r} = \mathbf{r}(t)$ be the characteristic curve passing through it, so that $\mathbf{r}(t_0) = \mathbf{r}_0$. Defining $U(t) := u(x(t), y(t)) - z(t)$, we have

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}t} &= u_x(x(t), y(t))\frac{\mathrm{d}x}{\mathrm{d}t} + u_y(x(t), y(t))\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{\mathrm{d}z}{\mathrm{d}t} \\
&= [u_x(x, y)a(x, y, z) + u_y(x, y)b(x, y, z) - c(x, y, z)]k(t) \\
&= [u_x(x, y)a(x, y, u(x, y) - U) + u_y(x, y)b(x, y, u(x, y) - U) \\
&\quad - c(x, y, u(x, y) - U)]k(t)
\end{aligned}$$

The last line, with $x = x(t)$ and $y = y(t)$ given by Γ, is an ODE in $U(t)$. Since $\mathbf{r}(t_0) \in \Sigma$, we have $U(t_0) = 0$, which serves as initial condition for the ODE. Substituting $U = 0$ into the right hand side of the ODE gives (5.3). Thus the zero function is a particular solution of the ODE. Since the ODE solution is unique, we have $U \equiv 0$, thus Γ is contained in Σ. ■

As a consequence of theorem 5.1, two integral surfaces that have a point $\mathbf{r}_0$ in common will intersect along the characteristic curve that passes through $\mathbf{r}_0$. The converse is also true:

Theorem 5.2 *The intersection curve of two integral surfaces is a characteristic curve.*

PROOF. Consider two integral surfaces that intersect along a curve Γ. By this we mean that the two surfaces have distinct normals along the curve Γ of common points. At any point of Γ, the surfaces' distinct tangent planes both have to contain $\mathbf{a}$. Since the intersection of the tangent planes is the tangent to Γ, Γ is tangential to $\mathbf{a}$, and so Γ is a characteristic curve. ■

Example 1 (continued)
Setting $C_3 = 1$ in the general solution for the characteristic curves gives the family of solutions

$$x/z = C_1 \quad y/z = C_2$$

This can be interpreted as the set of intersection curves of the integral surfaces $u_1(x, y) := x/C_1$ and $u_2(x, y) := y/C_2$. These integral surfaces are the level surfaces of the scalar fields $F_1(\mathbf{r}) := x/z$ and $F_2(\mathbf{r}) := y/z$.

5.1.2 Cauchy Problem

The *Cauchy problem* for the first-order quasilinear PDE (5.1) is to find the integral surface that contains a given smooth curve

$$\Gamma_0 : \mathbf{r} = \mathbf{f}(s) = [f(s), g(s), h(s)]$$

This is called the *initial curve,* and the equation

$$u(f(s), g(s)) = h(s)$$

is called the *initial condition*. In many applications y represents time t and the initial condition is written $u(s, t_0) = h(s)$. In this case the Cauchy problem is called an *initial value problem.*

The basic idea for solving the Cauchy problem is as follows. For every point $\mathbf{f}(s)$ on initial curve Γ_0, find the characteristic curve $\Gamma : \mathbf{r} = \mathbf{R}(t, s)$ that passes through the point. This is done by solving a vector ODE similar to (5.4), namely

$$\frac{\partial \mathbf{R}}{\partial t} = k(t)\mathbf{a}(\mathbf{R})$$

where k is an arbitrary continuous nonzero function. The ODE initial condition is $\mathbf{R}(t_0, s) = \mathbf{f}(s)$. The set of ODE solutions

$$\mathbf{R}(t, s) =: [X(t, s), Y(t, s), Z(t, s)]$$

defines a surface in space parametrised by s and t.

To find the integral surface in the form (5.2), we need to solve the base characteristic curves equations

$$x = X(t, s), \quad y = Y(t, s) \tag{5.7}$$

for t and s. This gives

$$t = T(x, y), \quad s = S(x, y)$$

The integral surface is then given by

$$u(x, y) = Z(T(x, y), S(x, y))$$

Example 1 (continued)
Continuing with the PDE $x\,u_x + y\,u_y = u$, consider the Cauchy problem with initial condition $u(s, 1) = h(s)$. The initial value ODE problem is

```
> sol:=dsolve({ODES,x(1)=s,y(1)=1,z(1)=h(s)},
>     {x(t),y(t),z(t)},explicit);
```

$$sol := \{\mathrm{x}(t) = t\,s,\ \mathrm{y}(t) = t,\ \mathrm{z}(t) = t\,\mathrm{h}(s)\}$$

Solving for z in terms of x and y gives

```
> solve(subs(z(t)=z,x(t)=x,y(t)=y,sol),{z,s,t});
```

$$\{t = y,\ z = y\,\mathrm{h}(\frac{x}{y}),\ s = \frac{x}{y}\}$$

Verify that $u(x,\ y) = yh(x/y)$ satisfies the PDE and the initial condition:

```
> U:=(x,y)-> y*h(x/y):
> is(x*diff(U(x,y),x)+y*diff(U(x,y),y)=U(x,y));
```

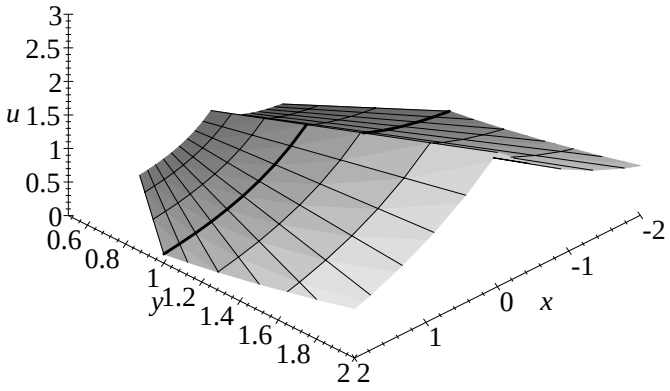
true

```
> is(U(s,1)=h(s));
```

true

The Maple function `PDEplot` plots the solution of the Cauchy Problem using numerical ODE integration algorithms such as the Runge-Kutta method. For example the solution when the initial condition is $u(s, 1) = e^{-|s|}$ is plotted as follows.

```
> with(DEtools):
> h:=s->piecewise(s<0,exp(s),s>=0,exp(-s)):
> PDEplot([x,y,u(x,y)],u(x,y),[s,1,h(s)],s=-2..2,
>   x=-2..2,y=1/2..2,u=0..3,numchar=19,style=patch);
```



Notice how the kink in $h(s) = e^{-|s|}$ at $s = 0$ is propagated along the characteristic curve $\{[0, t, t] | t \in \mathsf{R}\}$ that passes through $\mathbf{f}(0) = [0, 1, 1]$. Because of the kink, this is only a solution in the weak sense.

By the implicit function theorem, a solution for (5.7) exists in a neighborhood of a point (t_0, s_0) on the initial curve provided that the jacobian

$$D(t, s) := \det \begin{bmatrix} \dfrac{\partial X}{\partial t} & \dfrac{\partial X}{\partial s} \\ \dfrac{\partial Y}{\partial t} & \dfrac{\partial Y}{\partial s} \end{bmatrix}$$

is nonzero at that point. When $D(t_0, s_0) \neq 0$, the solutions $T(x, y)$ and $S(x, y)$ exist and are continuously differentiable in a neighborhood of the point $[f(s_0), g(s_0)]$.

At a point $s = s_0$ on the initial curve we have

$$D(t_0, s_0) = \det \begin{bmatrix} X_t(t_0, s_0) & X_s(t_0, s_0) \\ Y_t(t_0, s_0) & Y_s(t_0, s_0) \end{bmatrix} = \det \begin{bmatrix} k(t_0)a(\mathbf{f}(s_0)) & f'(s_0) \\ k(t_0)b(\mathbf{f}(s_0)) & g'(s_0) \end{bmatrix}$$

Then, the condition $D(t_0, s_0) \neq 0$ is equivalent to the condition that the vector $[a, b]^T$ is not parallel to vector $[f', g']^T$. This can be interpreted geometrically as requiring that the projection of the initial curve onto the xy plane is not tangential to the base characteristic curve.

If $D(t_0, s_0) = 0$, then the Cauchy problem has no solution or an infinity of solutions. This is the content of the following two theorems.

Theorem 5.3 *If $D(t_0, s_0) = 0$, then the Cauchy problem is not solvable unless the initial curve is tangential to the characteristic curve at $\mathbf{r}_0 := \mathbf{f}(s_0)$.*

PROOF. If $D(t_0, s_0) = 0$, $[f'(s_0), g'(s_0)]$ is parallel to $[a(\mathbf{r}_0), b(\mathbf{r}_0)]$, that is,

$$[f'(s_0), g'(s_0)] = \kappa [a(\mathbf{r}_0), b(\mathbf{r}_0)]$$

for some nonzero constant κ. Let $u(x, y)$ be a solution of the Cauchy problem for the quasilinear PDE (5.1). Differentiating the initial condition

$$h(s) = u(f(s), g(s))$$

with respect to s gives

$$h'(s) = u_x(f(s), g(s))\, f'(s) + u_y(f(s), g(s))g'(s)$$

At (s_0, t_0) this becomes

$$\begin{aligned} h'(s_0) &= u_x(f(s_0), g(s_0))\, f'(s_0) + u_y(f(s_0), g(s_0))g'(s_0) \\ &= u_x(f(s_0), g(s_0))[\kappa a(\mathbf{r}_0)] + u_y(f(s_0), g(s_0))[\kappa b(\mathbf{r}_0)] \\ &= \kappa c(\mathbf{r}_0) \end{aligned}$$

Thus

$$[f'(s_0), g'(s_0), h'(s_0)] = \kappa[a(\mathbf{r}_0), b(\mathbf{r}_0), c(\mathbf{r}_0)]$$

so that the initial curve is tangential to the characteristic curve. ■

Theorem 5.4 *If the initial curve is a characteristic curve then the Cauchy problem has an infinite number of solutions.*

PROOF. Choose any point $\mathbf{r}_0$ on the initial curve Γ_0, and take any smooth curve Γ_1 that includes $\mathbf{r}_0$ and that is not tangential to the characteristic curve Γ_0 at $\mathbf{r}_0$. Then the xy plane projection of Γ_1 is not tangential to the base characteristic curve at $\mathbf{r}_0$, so the Cauchy problem with initial curve Γ_1 has a solution. Since this integral surface includes the characteristic curve Γ_0, it also solves the original Cauchy problem. ■

Example 1 (continued)
Consider a Cauchy problem with the x-axis as initial base curve.

```
> h:='h': f:=s: g:=0:
> Gamma[0]:=[f,g,h(s)];
```

$$\Gamma_0 := [s, 0, \mathrm{h}(s)]$$

The condition for existence of a unique solution is that the following determinant is nonzero:

```
> J:=array([ [ a(f,g,h)[1], diff(f,s) ],
>            [ a(f,g,h)[2], diff(g,s) ]]);
```

$$J := \begin{bmatrix} s & 1 \\ 0 & 0 \end{bmatrix}$$

```
> linalg[det](J);
```

$$0$$

Since the determinant is zero, the Cauchy problem is only solvable if Γ_0 is tangential to the characteristic direction field, that is, the cross product of the curve tangent with the direction field should be zero.

```
> Tangent:=map(r->diff(r,s),Gamma[0]);
```

$$\mathit{Tangent} := \left[1,\ 0,\ \frac{\partial}{\partial s}\mathrm{h}(s)\right]$$

```
> DirectionField:=a(op(Gamma[0]));
```

$$\mathit{DirectionField} := [\,s,\ 0,\ \mathrm{h}(s)\,]$$

```
> linalg[crossprod](Tangent,DirectionField);
```

$$\left[0,\ \left(\frac{\partial}{\partial s}\mathrm{h}(s)\right) s - \mathrm{h}(s),\ 0\right]$$

Thus, for example, the initial condition $u(s,0) = 0$ is admissible, that is, $h \equiv 0$ and Γ_0 is the x-axis. To solve this Cauchy problem, we choose the following curve that passes through Γ_0:

```
> Gamma[1]:=[1,sigma,C*sigma];
```

$$\Gamma_1 := [1,\ \sigma,\ C\,\sigma]$$

This curve is not characteristic, since the cross product of its tangent with the characteristic direction vector is nonzero:

```
> Tangent:=map(r->diff(r,s),Gamma[1]);
```

$$\mathit{Tangent} := [\,0,\ 1,\ C\,]$$

```
> DirectionField:=a(op(Gamma[1]));
```

$$\mathit{DirectionField} := [\,1,\ \sigma,\ C\sigma\,]$$

```
> linalg[crossprod](Tangent,DirectionField);
```

$$[\,0,\ C,\ -1\,]$$

Solving the initial value ODE problem for the Cauchy problem with initial curve Γ_1 gives

```
> sol:=dsolve({ODES,x(1)=1,y(1)=sigma,z(1)=C*sigma},
> {x(t),y(t),z(t)},explicit);
```

$$\mathit{sol} := \{\mathrm{y}(t) = t\,\sigma,\ \mathrm{z}(t) = t\,C\,\sigma,\ \mathrm{x}(t) = t\}$$

The integral surface is found by solving for z as a function of x and y:

```
> solve(subs(z(t)=z,x(t)=x,y(t)=y,sol),{z,sigma,t});
```

$$\{t = x,\ z = C\,y,\ \sigma = \frac{y}{x}\}$$

There are infinitely many solutions of the form $u(x,y) = Cy$, one solution for every value of C. Verify that they all satisfy the PDE and the initial condition:

```
> U:=(x,y)-> C*y:
> is(x*diff(U(x,y),x)+y*diff(U(x,y),y)=U(x,y))
> and is(U(s,1)=0);
```

true

5.2 Single Quasilinear First Order PDE in n Independent Variables

5.2.1 Generalisation to n Independent Variables

The generalisation of results of the preceding section to more than two independent variables is essentially just a matter of notation. A quasilinear first order PDE for a function $u(\mathbf{x})$ of n variables has the form

$$\mathbf{a}(\mathbf{x}, u) \cdot u_{\mathbf{x}} = c(\mathbf{x}, u)$$

with $\mathbf{a}$ nowhere zero.

The Cauchy problem is the search for an integral surface

$$z = u(\mathbf{x})$$

that contains the initial $n-1$-dimensional manifold

$$\Gamma_0 : \mathbf{r} = \mathbf{f}(\mathbf{s})$$

where the parameter $\mathbf{s}$ ranges over some domain in R^{n-1}. The Cauchy problem solution is constructed by first solving the characteristic ODE system

$$\frac{\partial \mathbf{R}}{\partial t} = k(t) \begin{bmatrix} \mathbf{a}(\mathbf{R}) \\ c(\mathbf{R}) \end{bmatrix}$$

with initial condition

$$\mathbf{R}(t_0, \mathbf{s}) = \begin{bmatrix} \mathbf{f}(\mathbf{s}) \\ h(\mathbf{s}) \end{bmatrix}$$

The ODE solution is

$$\mathbf{R}(t, \mathbf{s}) =: \begin{bmatrix} \mathbf{X}(t, \mathbf{s}) \\ Z(t, \mathbf{s}) \end{bmatrix}$$

If the jacobian

$$D(t_0, \mathbf{s}_0) := \det \begin{bmatrix} k(t_0)\mathbf{a}(\mathbf{f}(\mathbf{s}_0), h(\mathbf{s}_0)) & \mathbf{f}_{\mathbf{s}}(\mathbf{s}_0) \end{bmatrix}$$

is nonzero, then the equations $\mathbf{x} = \mathbf{X}(t, \mathbf{s})$ can be solved (at least locally) for

$$t = T(\mathbf{x}), \ \ \mathbf{s} = \mathbf{S}(\mathbf{x})$$

and the integral surface is given by

$$u(\mathbf{x}) = Z(T(\mathbf{x}), \mathbf{S}(\mathbf{x}))$$

5.2.2 Conservation Laws

The *conservation law* is an important example of a quasilinear PDE in several variables. Let the scalar field $u(\mathbf{x}, t)$ represent the density of some substance in a domain Ω, and let the vector field $\mathbf{q}$ represent the flux density, given in units of substance per unit volume per unit time. If there are no sources or sinks then any decrease in the amount of substance must be accounted for by a flow out through the boundary of the domain. Since the total amount is $\int_\Omega u \, \mathrm{d}V$ while the net rate of flow out of the domain is $\int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}S$, this requirement can be written

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u \, \mathrm{d}V = \int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}S$$

Assuming that the domain Ω does not vary in time, the time differentiation can be carried into the integral. Also, the divergence theorem can be applied to the right hand side. This gives the integral equation

$$\int_\Omega \left(\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} \right) \mathrm{d}V = 0 \tag{5.8}$$

Since Ω can be any source-free volume we must have

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = 0$$

This is the conservation law. When the flux $\mathbf{q}$ is a given function of density u, the conservation law gives the first order quasilinear PDE

$$u_t + \mathbf{q}'(u) \cdot u_{\mathbf{x}} = 0 \tag{5.9}$$

where $\mathbf{q}' := \mathrm{d}\mathbf{q}/\mathrm{d}u$.

The ODEs for the characteristic equations with $k \equiv 1$ are

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \mathbf{q}'(z), \ \frac{\mathrm{d}t}{\mathrm{d}\tau} = 1, \ \frac{\mathrm{d}z}{\mathrm{d}\tau} = 0$$

The ODE initial conditions corresponding to the PDE Cauchy problem initial condition $u(\mathbf{s}, 0) = h(\mathbf{s})$ are

$$\mathbf{x}(0) = \mathbf{s}, \ t(0) = 0, \ z(0) = h(\mathbf{s})$$

Solving the characteristic ODEs gives

$$\mathbf{x} = \mathbf{q}'(z)\tau + \mathbf{s} \quad t = \tau, \quad z = h(\mathbf{s})$$

Combining these gives

$$z = h(\mathbf{x} - t\mathbf{q}'(z))$$

Solving this for z gives the integral (hyper-)surface $z = u(\mathbf{x}, t)$.

To verify that this solution is correct, first differentiate both sides of

$$u = h(\mathbf{x} - t\mathbf{q}'(u)) \tag{5.10}$$

with respect to t:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} \frac{\partial}{\partial t}\left[x_j - tq_j'(u)\right] \\
&= \sum_{j=1}^{n} \frac{\partial h}{\partial x_j}\left[-q_j'(u) - tq_j'' u_t\right] \\
&= -h_{\mathbf{x}} \cdot \mathbf{q}' - tu_t h_{\mathbf{x}} \cdot \mathbf{q}''
\end{aligned}$$

which can be solved to give

$$u_t = \frac{-h_{\mathbf{x}} \cdot \mathbf{q}'}{1 + th_{\mathbf{x}} \cdot \mathbf{q}''} \tag{5.11}$$

Differentiating (5.10) with respect to x_i gives

$$\begin{aligned}
\frac{\partial u}{\partial x_i} &= \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} \frac{\partial}{\partial x_i}\left[x_j - tq_j'(u)\right] \\
&= \sum_{j=1}^{n} \frac{\partial h}{\partial x_j}\left[\delta_{ij} - tq_j'' u_i\right] \\
&= h_i - tu_i \sum_{j=1}^{n} h_j q_j''
\end{aligned}$$

This can be written in vector form as

$$u_{\mathbf{x}} = h_{\mathbf{x}} - t(h_{\mathbf{x}} \cdot \mathbf{q}'')u_{\mathbf{x}}$$

Solving gives

$$u_{\mathbf{x}} = \frac{h_{\mathbf{x}}}{1 + th_{\mathbf{x}} \cdot \mathbf{q}''} \tag{5.12}$$

Equations (5.11) and (5.12) show that the PDE (5.9) is satisfied provided that the denominators do not vanish.

If the denominator in (5.11) and (5.12) does vanish, then a singularity appears in the solution. These singularities signal the onset of *shocks*, which are regions where the integral surface ceases to be single valued as a function of $\begin{bmatrix} \mathbf{x} \\ t \end{bmatrix}$, although it continues to be a single valued function of $\begin{bmatrix} \mathbf{x} \\ t \\ z \end{bmatrix}$. The shock first appears at the time

$$t_{\text{critical}} = \min_{\mathbf{s}} \frac{-1}{\mathbf{q}''(h(\mathbf{s})) \cdot h_{\mathbf{s}}(\mathbf{s})}$$

This kind of loss of solution uniqueness is called a *gradient catastrophe*.

To model the solution in the shock region, the integral form of the conservation law (5.8) is used instead of the PDE, leading to so-called *weak solutions*. This is beyond the scope of this course.

Example 2
In a classic continuum model for the flow of traffic in a one-lane highway, the density (number of cars per unit distance) is denoted $u(x,t)$ and the flow rate (number of cars per unit time) is denoted $q(u)$. The conservation law is

```
> PDE:=diff(u(x,t),t)+diff(q(u(x,t)),x)=0;
```

$$PDE := \left(\frac{\partial}{\partial t}\,\mathrm{u}(x,\,t)\right) + \mathrm{D}(q)(\mathrm{u}(x,\,t))\,\left(\frac{\partial}{\partial x}\,\mathrm{u}(x,\,t)\right) = 0$$

A simple model for the traffic is to assume that the speed decreases linearly with the density until, at some critical density u_{jam}, the speed is zero. The flow rate is the product of density and speed.

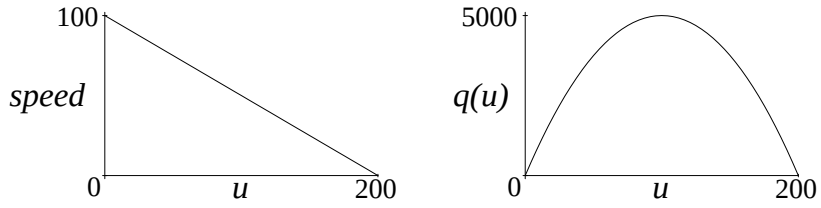
```
> speed:=v_max*(1-u/u_jam);
```

$$speed := v\_max\,(1 - \frac{u}{u\_jam})$$

```
> q:=unapply(speed*u,u);
```

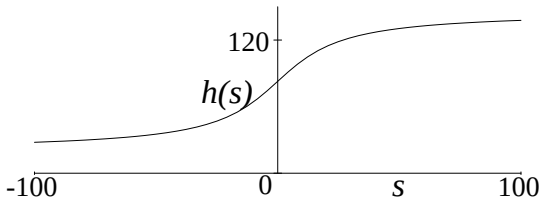
$$q := u \to v\_max\,(1 - \frac{u}{u\_jam})\,u$$

```
> v_max:=100:   u_jam:=200:
> plot(speed,u=0..200,labels=[u,'speed']);
> plot(q(u),u=0..200,labels=[u,'q(u)']);
```



Suppose that the initial car density is given by the following profile, which describes a transition from low density to high density.

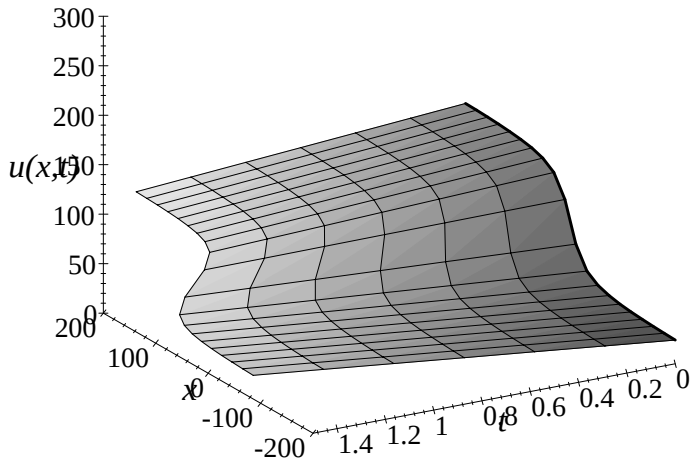
```
> h:=s->20+40*(arctan(s/20)+Pi/2):
> plot(h,-100..100,view=[-100..100,0..150],
>      labels=['s','h(s)']);
```



Using `PDEplot` to solve the initial value problem, we see that the solution

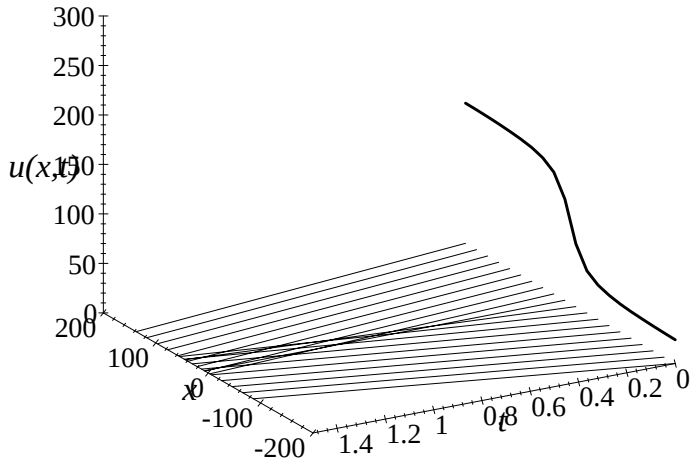
becomes multiple valued.

```
> with(DEtools):
> PDEplot(PDE,u(x,t),[s,0,h(s)],s=-200..200,t=0..1.5,
>  x=-200..200,u=0..300,style=patch,
>     orientation=[150,65]);
```



We can see more clearly where the shock develops when we look at the base characteristic curves.

```
> PDEplot(PDE,u(x,t),[s,0,h(s)],s=-200..200,t=0..1.5,
>  x=-200..200,u=0..300,style=patch,
>  orientation=[150,65],basechar=only);
```



The shock begins at

```
> t_critical:=minimize(-1/(D(h)(s)*(D@@2)(q)(h(s))));
```

$$t\_critical := \frac{1}{2}$$

5.3 Systems of First Order PDEs

5.3.1 Notation and Classification

A system of *quasilinear* partial differential equations in the m dependent functions $\mathbf{u}$ of n independent variables $\mathbf{x}$ has the form

$$\sum_{i=1}^{n} \mathbf{A}_i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}(\mathbf{x})}{\partial x_i} = \mathbf{c}(\mathbf{x}, \mathbf{u}) \tag{5.13}$$

where the $m \times m$ matrix $\mathbf{A}_i$ and m-vector $\mathbf{c}$ have continuously differentiable elements in some domain $\Omega \subset \mathsf{R}^{m+n}$. The left hand side is called the *principal part* of the system. If the coefficients $\mathbf{A}_i$ do not depend on $\mathbf{u}$ then the system is called *almost linear*. If in addition $\mathbf{c}$ is linear in $\mathbf{u}$ then the system is called *linear*.

Just as in ODE theory, a single quasilinear PDE of high order can be reduced to a system of first order PDEs by introducing new variables. The following example illustrates this.

Example 3
The damped one-dimensional wave equation

$$w_{xx} - e\, w_t - w_{tt} = 0$$

can be converted into a system of two first-order PDEs by introducing the variables $u_1 = w_x$, $u_2 = w_t$, so that the wave equation is

$$\begin{aligned} \frac{\partial}{\partial t} u_1 - \frac{\partial}{\partial x} u_2 &= 0 \\ \frac{\partial}{\partial t} u_2 - \frac{\partial}{\partial x} u_1 &= -e\, u_2 \end{aligned}$$

This can be written as a system of the form (5.13) with $\mathbf{I}$ as the coefficient of $\mathbf{u}_t$,

```
> A:=matrix(2,2,[[0,-1],[-1,0]]);
```

$$A := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

as the coefficient of $\mathbf{u}_x$, and

```
> c:=vector(2,[0,-e*u[2]]);
```

$$c := [0,\ -e\, u_2]$$

An $n-1$-dimensional manifold Γ is said to be a *base characteristic* if the partial derivatives of $\mathbf{u}$ are not determined uniquely from the data on the manifold and the PDE (5.13). This occurs whenever the manifold's normal vector $\boldsymbol{\beta}$ is such that $\sum_{i=1}^{n} \beta_i \mathbf{A}_i$ is singular, so that the homogeneous matrix equation

$$\left(\sum_{i=1}^{n} \beta_i \mathbf{A}_i \right) \mathbf{p} = 0 \tag{5.14}$$

admits a nontrivial solution $\mathbf{p}$. For then if $\{\mathbf{u}_{x_i}\}$ is a solution of (5.13), so is $\{\mathbf{u}_{x_i} + \alpha\beta_i \mathbf{p}\}$, for any constant α.

This definition of a characteristic manifold is consistent with the definition of earlier sections. There, a base characteristic manifold was defined as one whose tangent $\mathrm{d}\mathbf{x}/\mathrm{d}t$ is parallel to the vector field $\mathbf{a}$, that is,

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = k(t)\mathbf{a}(t)$$

Since the manifold's normal is orthogonal to the tangents, we have

$$\boldsymbol{\beta} \cdot \mathrm{d}\mathbf{x} = \boldsymbol{\beta} \cdot (k\mathbf{a}\mathrm{d}t) = \left(\sum_{i=1}^{n} \beta_i a_i \right) k\mathrm{d}t = 0$$

and this is the form of (5.14) when the system consists of a single PDE.

When time is involved it is natural to write the system of quasilinear first order PDEs in the form

$$\mathbf{B}(\mathbf{x}, t, \mathbf{u}) \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} + \sum_{i=1}^{n} \mathbf{A}_i(\mathbf{x}, t, \mathbf{u}) \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial x_i} = \mathbf{c}(\mathbf{x}, t, \mathbf{u}) \tag{5.15}$$

instead of the form (5.13). Henceforth we shall only be dealing with this type of PDE system.

The base characteristics for (5.15) are determined by an equation similar to (5.14), which can be written in the form

$$\lambda \mathbf{B}\mathbf{p} = \left(\sum_{i=1}^{n} \beta_i \mathbf{A}_i \right) \mathbf{p} \tag{5.16}$$

This is a generalised eigenvalue problem for an eigenvalue λ and corresponding eigenvector $\mathbf{p}$. The tangent to the base characteristic manifold then satisfies

$$\boldsymbol{\beta} \cdot \mathrm{d}\mathbf{x} = \lambda \mathrm{d}t \tag{5.17}$$

The system (5.15) is said to be *hyperbolic* at a point $(\mathbf{x}, t, \mathbf{u})$ if $\mathbf{B}$ is nonsingular and if, for any nonzero assignment of the real parameters $\beta_1, \ldots, \beta_n$, the generalised eigenvalue problem (5.16) has m real eigenvalues and m real linearly independent eigenvectors. A hyperbolic PDE system is thus characterised by having a full set of m real distinct base characteristics. In particular, if there are m distinct eigenvalues then there is a full set of m real linearly independent eigenvectors.

The single quasilinear first-order PDEs studied in the previous sections are hyperbolic. To show this, assume $a_1 \neq 0$ and rename $a_1 =: b$, $x_1 =: t$; the eigenvalue is then

$$\lambda = \frac{\beta_2 a_2 + \cdots + \beta_n a_n}{b}$$

and the eigenvector is 1.

The stipulation that **B** be nonsingular ensures that Cauchy problems with initial conditions of the form

$$\mathbf{u}(\mathbf{x}, t_0) = \mathbf{h}(\mathbf{x})$$

are solvable. The initial manifold here is $\Gamma_0 : t = t_0$. The normals to this manifold are of the form

$$\begin{bmatrix} \lambda \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$$

Substituting this into (5.16) gives

$$\mathbf{Bp} = 0$$

which has no nontrivial solution. Thus this initial manifold is not a base characteristic.

For a hyperbolic system there exists a real diagonal matrix $\boldsymbol{\Lambda}$ and a nonsingular matrix **P** such that the diagonalising decomposition

$$\left(\sum_{i=0}^{n} \beta_i \mathbf{A}_i\right) \mathbf{P} = \mathbf{BP}\boldsymbol{\Lambda}$$

is possible. The diagonal elements of $\boldsymbol{\Lambda}$ are simply the eigenvalues of the generalised eigenvalue problem (5.16) and **P** is a matrix whose columns are the corresponding eigenvectors.

Example 3 (continued)
The eigenvalues of **A** for the one-dimensional wave equation are

```
> with(linalg):
> eigenvals(A);
```

$$-1,\ 1$$

Since the eigenvalues are real and distinct, the system is hyperbolic.

A diagonalising decomposition is given by

```
> eA:=eigenvects(A);
```

$$eA := [-1,\ 1,\ \{[1,\ 1]\}],\ [1,\ 1,\ \{[-1,\ 1]\}$$

```
> P:=transpose(matrix(2,2,[seq(op(eA[i][3]),
>        i=1..2)]));
```

$$P := \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Verify that this change of variables diagonalises the coefficient matrix.

```
> Lambda:=linsolve(P,multiply(A,P));
```

$$\Lambda := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

5.3.2 Canonical Form of Hyperbolic Systems in Two Independent Variables

We now focus on first order hyperbolic PDE systems in two independent variables, that is, one spatial variable x and time t. Without loss of generality we can take $\mathbf{B} = \mathbf{I}$, so that the system has the form

$$\mathbf{u}_t(x,t) + \mathbf{A}(x,t,\mathbf{u})\mathbf{u}_x(x,t) = \mathbf{c}(x,t,\mathbf{u}) \tag{5.18}$$

For simplicity, we shall only consider hyperbolic systems in which $\mathbf{A}$ has m distinct real eigenvalues in some domain $\Omega \in \mathsf{R}^{m+2}$. Then $\mathbf{A}$ has the diagonalising decomposition

$$\mathbf{AP} = \mathbf{P\Lambda}$$

where $\mathbf{P}(x,t,\mathbf{u})$ is a real orthogonal matrix of eigenvectors and $\mathbf{\Lambda}(x,t,\mathbf{u})$ is a diagonal matrix of eigenvalues.

Introducing the new variables

$$\mathbf{v} := \mathbf{P}^{-1}\mathbf{u}$$

into (5.18) gives

$$\mathbf{P}\mathbf{v}_t + \mathbf{P}_t\mathbf{v} + \mathbf{A}(\mathbf{P}\mathbf{v}_x + \mathbf{P}_x\mathbf{v}) = \mathbf{c}$$

Premultiplying both sides by $\mathbf{P}^{-1}$ and introducing $\mathbf{d} := \mathbf{P}^{-1}(\mathbf{c} - \mathbf{P}_t\mathbf{v} - \mathbf{A}\mathbf{P}_x\mathbf{v})$ gives

$$\mathbf{v}_t + \mathbf{\Lambda}\mathbf{v}_x = \mathbf{d}(x,t,\mathbf{v}) \tag{5.19}$$

This is the *canonical form* of the PDE system (5.18), in which the dependent variables in the principal part are decoupled.

Example 3 (continued)
Introduce new variables to bring the first order PDE system for the one-dimensional wave equation into canonical form.

```
> u:=vector(2):
> v=linsolve(P,u);
```

$$v = \left[\frac{1}{2}u_1 + \frac{1}{2}u_2,\ -\frac{1}{2}u_1 + \frac{1}{2}u_2\right]$$

```
> v:=vector(2):
> u:=evalm(P&*v);
```

$$u := [v_1 - v_2,\ v_1 + v_2]$$

```
> d:=linsolve(P,c);
```

$$d := \left[-\frac{1}{2}\,e\,(v_1 + v_2),\ -\frac{1}{2}\,e\,(v_1 + v_2)\right]$$

```
> for i from 1 to 2 do
>  PDE[i]:=Diff(v[i],t)+Lambda[i,i]*Diff(v[i],x)=d[i]
> od;
```

$$PDE_1 := \left(\frac{\partial}{\partial t}\,v_1\right) - \left(\frac{\partial}{\partial x}\,v_1\right) = -\frac{1}{2}\,e\,(v_1 + v_2)$$

$$PDE_2 := \left(\frac{\partial}{\partial t}\,v_2\right) + \left(\frac{\partial}{\partial x}\,v_2\right) = -\frac{1}{2}\,e\,(v_1 + v_2)$$

Base characteristic curves for (5.18) can be described by equations of the form $x = f_j(t)$, since curves parallel to the x-axis are not characteristic. The m base characteristic curves are supposed to satisfy (5.17), which for two independent variables reduces to

$$\frac{\mathrm{d}f_j}{\mathrm{d}t} = \lambda_j \quad (j = 1, \ldots, m)$$

If the system is almost linear, then the base characteristic directions λ_j do not depend on the solution $\mathbf{u}$. For general quasilinear PDE systems, however, the base characteristics are solution dependent.

The base characteristics can be used to transform a hyperbolic PDE system into a system of ordinary differential equations, as follows. Let $V_j(t) := v_j(f_j(t), t)$ be the value of the jth canonical solution variable along a base characteristic. Then differentiating along the base characteristic gives

$$\begin{aligned}\frac{\mathrm{d}V_j}{\mathrm{d}t} &= \frac{\partial v_j}{\partial x}\frac{\mathrm{d}f_j}{\mathrm{d}t} + \frac{\partial v_j}{\partial t} \\ &= \frac{\partial v_j}{\partial x}\lambda_j + \frac{\partial v_j}{\partial t} \\ &= d_j(f_j(t), t, \mathbf{v}(f_j(t), t))\end{aligned}$$

These ODEs are used in the *method of characteristics*. When the original PDEs is homogeneous and has constant coefficients, the base characteristic directions are constant, and the canonical variables are constant along their characteristics. The method of characteristics can then be used to find an exact solution. This is illustrated in the following example.

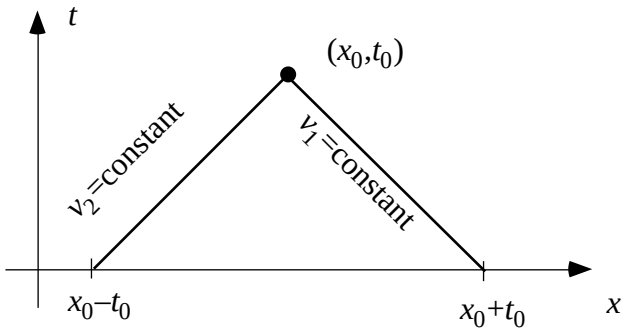


Figure 5.1: Base characteristics of one-dimensional wave equation.

Example 3 (continued)
For the one dimensional wave equation the two characteristic directions are $\lambda = \pm 1$, so the base characteristics passing through (x_0, t_0) are the straight lines

```
> dsolve({diff(f[1](t),t)=Lambda[1,1],f[1](t0)=x0},
>            f[1](t));
```

$$f_1(t) = -t + t0 + x0$$

```
> dsolve({diff(f[2](t),t)=Lambda[2,2],f[2](t0)=x0},
>            f[2](t));
```

$$f_2(t) = t - t0 + x0$$

If there is no damping ($e \equiv 0$), the canonical equation with $V_1 := v_1(f_1(t), t)$ gives

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = 0$$

so that V_1 is constant along the base characteristic $x = f_1(t)$ (Figure 5.1). Similarly, $V_2 := v_2(f_2(t), t)$ is constant along the base characteristic $x = f_2(t)$.

If the initial values are given in the form

$$u(x, 0) = k(x), \quad u_t(x, 0) = l(x)$$

then we have

$$\begin{aligned} v_1(x_0, t_0) &= v_1(x_0 + t_0, 0) \\ &= \frac{1}{2}[u_1(x_0 + t_0, 0) + u_2(x_0 + t_0, 0)] \\ &= \frac{1}{2}[k_x(x_0 + t_0) + l(x_0 + t_0)] \end{aligned}$$

and similarly

$$v_2(x_0, t_0) = \frac{1}{2}[-k_x(x_0 - t_0) + l(x_0 - t_0)]$$

The solution at (x_0, t_0) is then

$$\begin{aligned} w_x(x_0, t_0) &= u_1(x_0, t_0) \\ &= v_1(x_0, t_0) - v_2(x_0, t_0) \\ &= \frac{1}{2}[k_x(x_0 + t_0) + k_x(x_0 - t_0)] + \frac{1}{2}[l(x_0 + t_0) - l(x_0 - t_0)] \end{aligned}$$

and similarly

$$w_t(x_0, t_0) = \frac{1}{2}[k_x(x_0 + t_0) - k_x(x_0 - t_0)] + \frac{1}{2}[l(x_0 + t_0) + l(x_0 - t_0)]$$

This agrees with the d'Alembert's formula for the solution of an undamped one dimensional wave equation presented in chapter 4.

For general quasilinear PDE systems, the method of characteristics is a numerical method for integrating along the base characteristics. However, because of difficulty of dealing base characteristics that are manifolds, the method is not widely used for problems with more than two independent variables. The treatment of shocks is also complicated in the method of characteristics, especially in higher dimensional problems.

Exercises

1. Find the characteristic curves for the following PDEs.

   (a) $xu_x + y^2u_y = u^2$

   (b) $xu_x = 1 + xu$

   (c) $uu_x + (y - u)u_y = 1 + u$

   (d) $u_x = xyu$

2. Multiplying both sides of the quasilinear PDE (5.1) by a nonzero function $d(x, y, u)$ doesn't change the PDE. Show that it also doesn't change the characteristic curves, only their parametrisation changes.

3. Transform the PDE in Example 1 to polar coordinates and find an expression for the family of characteristic curves. The family of curves should be the same as in Example 1.

4. Explain why the base characteristic curves of almost linear first order PDEs don't intersect.

5. Show that the characteristic curves of

   $$(x^2 - y^2)u_x + 2xyu_y = 0$$

   can be defined as the intersections of the two families of surfaces

   $$x^2 + y^2 = 2C_1y, \quad z = C_2$$

   where C_1 and C_2 are arbitrary constants. Describe these surfaces and curves geometrically.

6. Solve the following Cauchy problems. Plot solutions with `PDEplot`.

   (a) $u_y + \alpha u_x = 0,\ u(x, 0) = h(x)$ (α is a constant.)

   (b) $yu_x + xu_y = u^2,\ u(x, 0) = e^x$

   (c) $u_x - uu_y = 0,\ u(x, 0) = x^2$

   (d) $uu_x + uu_y = 1,\ u(\sin s \cos s, \cos^2 s) = \sin s$

7. An equation for water depth (measured relative to the rest state) in a narrow shallow canal is given by the conservation law

   $$\left(1 + \frac{3}{2}u\right)u_x + u_t = 0$$

   Solve the Cauchy problem with initial condition

   $$u(x, 0) = \begin{cases} \epsilon(1 + \cos x) & -\pi \le x \le \pi \\ 0 & \text{otherwise} \end{cases}$$

   where $0 < \epsilon \ll 1$ is a constant. At what time does the solution cease to exist? Use `PDEplot` to plot the base characteristic curves of a typical solution.

8. If $\mathbf{v}(\mathbf{x}, t)$ represents a velocity field then the acceleration field is given by the *convective derivative* (see [5]) of $\mathbf{v}$:

$$\frac{\mathrm{D}}{\mathrm{Dt}}\mathbf{v} := \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v}$$

The velocity field of a set of particles moving in one dimension with zero acceleration is therefore described by the PDE

$$v_t + vv_x = 0$$

Use `PDEplot` to plot the base characteristic curves for various initial velocity profiles such as $h(x) = \alpha x + \beta$, $h(x) = e^{\alpha x}$, $h(x) = \sin x$, and find the value of t when a shock develops. The interpretation is that when fast particles overtake slower ones, there is a collision.

9. Write the damped one dimensional wave equation

$$au_{xx} + bu_x + c - eu_t - u_{tt} = d$$

where all coefficients are functions of x and t and $a > 0$, as a system of first order PDEs of the form (5.18), and verify that the system is hyperbolic.

10. Write the equations for one dimensional inviscid isentropic gas flow

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0\\
\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0\\
\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + c^2\rho\frac{\partial u}{\partial x} &= 0
\end{aligned}$$

as a system of first order PDEs of the form (5.18), and verify that the system is hyperbolic.

11. Write the Maxwell equations

$$\mathbf{B}_t + c\nabla\times\mathbf{E} = 0, \quad \mathbf{E}_t - c\nabla\times\mathbf{B} = 0$$

for time-dependent vector fields $\mathbf{B}$ and $\mathbf{E}$ as a system of first order PDEs of the form (5.18), and verify that the system is hyperbolic.

12. The PDEs for one dimensional inviscid isentropic flow are given by

$$\begin{aligned}
u_t + uu_x + \frac{c(\rho)^2}{\rho}\rho &= 0\\
\rho_t + \rho u_x + u\rho_x &= 0
\end{aligned}$$

where $u(x,t)$ and $\rho(x,t)$ are the velocity and density of a gas in a pipe, and $c(\rho) > 0$ is a given function. Find the canonical form for this PDE system.

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