

PRODUCTS

Maple
MapleNet
Maple T.A.
Study Guide
Toolboxes
Third Party Products

CUSTOMER RESOURCES

Web Store
Application Center
MaplePrimes
Student Center
Maple for High Schools
MapleConnect
Training
Technical Support
Publications
Register Product

CORPORATE INFO

About Us
Press Room
Careers
Contact Us

Intro to Maple for PHYSICS Students

This tutorial was designed by Ross L. Spencer from Brigham Young University to introduce Maple to Physics students

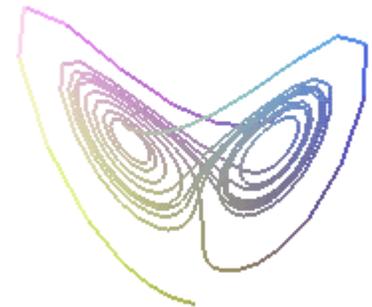
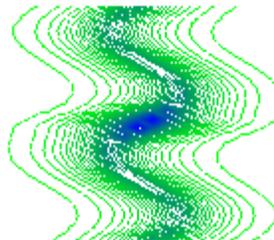
This course introduces the advantages of using Maple while studying physics. Several fields of mathematics are covered in this laboratory such as Calculus, Linear Algebra, Complex Numbers and many more. Each chapter is designed as a 3-hour laboratory period to walk you through each topic at your own speed. To allow students the ability to solve physics problems using Maple, this course offers a Laboratory Assignments worksheet with problems covering each chapter presented. A debugging worksheet accompanies the topics presented in this tutorial that provides a description of common problems users face when learning Maple for the first time.

Download the full tutorial

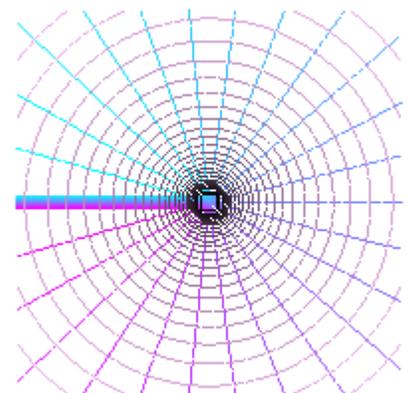
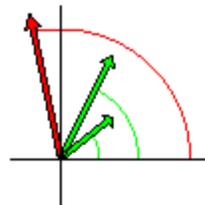
maple[®] **POWERTOOLS**

	preview	download
Introduction		
Chapter 1:		
Getting Started		
Chapter 2:		
Plotting		
Chapter 3:		
Calculus		
Chapter 4:		
Complex Numbers and Functions		

Chapter 2 shows the various plotting features and options available to create plots in Maple. Plotting gives the ability to visualize mathematical ideas that are important in physics.



Chapter 4 discusses complex arithmetic using Maple and provides visual representations to further explain this concept.



Chapter 5:

Linear Algebra  

Chapter 6:

Solving Equations  

Chapter 7:

Ordinary Differential Equations  

Chapter 8:

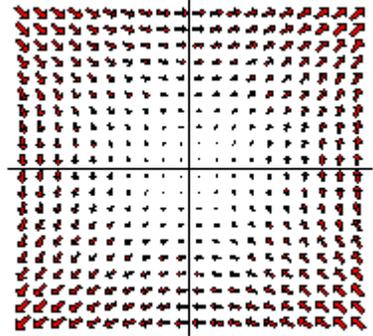
Programming -- logic, loops,
and procedures  

Chapter 9:

Symbolic Algebra  

Debugging  

Laboratory Assignments  



- [Introduction](#)
-

Introduction

This course is a laboratory in the use of the Maple computer mathematics program to do physics. With the advent of fast and cheap computers, programs like Maple will replace hand calculators and mathematical handbooks (like integral tables) for most physics students. Physics departments have already seen this happening in a rather random and unplanned way, so this course was invented to provide students with an introduction to the use of this powerful program. Once you learn it you will be hooked and you will wonder how you ever got along without it. This will especially be so if you don't just have Maple available on the department computers, but have it also on your own desktop computer, or even better, on your laptop. The time is rapidly approaching when laptops for physics students will be as common as slide rules were in olden times. If there is any way to pull it off, you ought to get a laptop .

Here's how the course will run. You and a lab partner will work through the chapters in this notebook at your own speed, assisted by the instructor and the TAs. The assigned problems for each 3-hour laboratory period are given in the last chapter of this book ([Laboratory assignments](#)). The only lectures will be answers to questions that are a little involved. There are no exams and no written homework. When you have finished an assigned problem, call a TA over and show them what you have done. Then go on to the next one. For Physics 230 there are three term projects required, but they are relatively simple: I want you to work three *significant* homework problems from math or physics courses that you are taking using a Maple worksheet. It would be a good idea to check with a TA or the instructor ahead of time to see if we think the problem is significant. You will get extra points for degree of difficulty.

Special note: *do not do your homework in this notebook* . Every once in a while I will make changes to this notebook and give you a new copy to use. The new edition will be worthless to you if all of your hard work is in the old version. So do all of your work in another file and leave this one the way you got it. A nice way to handle this is to have both this worksheet and your problem worksheet open simultaneously; then use the Tile, Horizontal, or Vertical option under Window on the toolbar (you get to choose which you like the best). Then you can read this worksheet and your problem worksheet on the same screen. This makes copying with the mouse from here to your assignment especially easy and you can read about what should be happening in the text while seeing what is actually happening in your own work.

You will be glad if you save your solutions to the problems in this text. You will find that your worked examples from this course will be of great value when you encounter similar problems in later courses.

This is a rather large piece of computer text and it is easy to lose track of things you might see once and want to find later. To help with this problem there is an index of hyperlinks to Maple topics in the next section. In addition you can find specific words or phrases in this text by using the Find selection under the Edit item on the toolbar.

And when things go wrong there are two debugging sections in this text. One is in Chapter 1 where debugging is introduced. The second is after Chapter 9 where specific errors and their cures are illustrated and discussed. To help you quickly go to this second debugging section there is a hyperlink to it in almost every subsection of the worksheet. And to help you go back to where you came from there is a set of chapter hyperlinks at the top of the debugging section. So if you need to quickly go to another chapter, first go to the debugging section, then click on the desired chapter hyperlink. Here, I'll show you. Click here on [Debugging](#) to go to the debugging section .

Finally, please contact me about errors in the text or trouble with Maple. The text errors I will fix and the Maple troubles will be added to the Debugging section at the end of the book. My email address is Ross_Spencer@byu.edu.

>

[Chapter1.mws](#)

- [Execution groups and paragraphs](#)
 - [Quick guide to managing the worksheet](#)
 - [Online help](#)
 - [Special constants](#)
 - [Variable names](#)
 - [What can a variable represent?](#)
 - [Complex numbers](#)
 - [Equations](#)
 - [Floating point numbers](#)
 - [Expressions and functions](#)
 - [Integers](#)
 - [Matrices](#)
 - [Polynomials](#)
 - [Strings](#)
 - [Vectors](#)
 - [Lists and sets](#)
 - [Getting the answer](#)
 - [Debugging \(when bad things happen to good programmers\)](#)
 - [evalf \(look at the numbers\)](#)
 - [check := vs =](#)
 - [plot \(make a picture\)](#)
 - [assume \(tie Maple down\)](#)
 - [restart \(punt\)](#)
 - [Cool stuff you can do right now](#)
 - [Integration](#)
 - [Differentiation](#)
 - [Build a sequence](#)
 - [Sum a series](#)
 - [Taylor series](#)
 - [Plot functions](#)
 - [Solve hard equations](#)
 - [Solve differential equations](#)
 - [Handle matrices and solve systems of equations](#)
 - [Do algebra](#)
-

Chapter 1: Getting started

Click on each subsection below, read and execute the Maple commands, and do the problems at the end of each one. There is a hyperlink at the top of each subsection called Debugging that will take you to the section at the end of the worksheet that discusses what errors mean and how to fix them.

Execution groups and paragraphs

[Debugging](#)

The first thing you will notice when you turn Maple on is the square bracket on the far left and the > sign just inside it. These two symbols are the most basic parts of Maple's graphical user interface and they will drive you crazy if you don't learn to handle them well right from the start. The window space marked off by the the square bracket is called an *execution group* and everything happens inside these groups. Here's what you have to know: everything before the > sign is text, or typeset mathematics, and the red stuff that comes after is Maple code that will execute to solve your problems. Maple results print in blue, like this:

Sine function, $\sin\left(\frac{\pi}{4}\right)$

> **sin(Pi/4);**

$$\frac{1}{2}\sqrt{2}$$

The purpose of the text portion is to document what you are doing. **Please document your code.** I will be glad you did, the grader will be glad you did, and six months from now when you remember that you once solved a problem like the one you just got assigned and you go looking for it, you will be glad you did.

>

Within execution groups are subunits called *paragraphs*. A paragraph might be a section of text, a Maple command, a Maple result, or maybe stuff neither one of us knows about yet. It's a little complicated. The easiest way to see what paragraphs are is to click on the little paragraph icon (backwards "p"-looking thing on the tool bar). This will print the special characters in the worksheet and will show you where the paragraphs are. Do it now and leave it on for a while until you get a feel for what paragraphs are.

Now hang in there, you're about to actually do something. There are two execution groups just below

this section of text. Put text that says **Kindergarten stuff** before the > sign in the first one (move the cursor just to the left of the > sign by using the arrow keys), hit Enter; then put 1+1 after the > sign and hit Enter again

>

You should now be looking at a mess inside the execution group; 1+1 didn't work and you got an error.

First things first, and this is important: the error occurred because Maple requires every executable statement *to end with a semicolon or a colon* . If you end with ; then the line will execute and Maple will tell you what it did. If you end with : then the line will execute, but Maple will not display the result. This is very useful if you know the result is OK but it's such a big mess that you don't want to see it.

Anyway, to get 1+1 to work you need to type **1+1;** to get the usual answer. But first you had better clean up the mess. The error line is a paragraph. If you try to delete it with BkSp or Del, nothing will happen because these are lines that Maple generated. To kill it you need the Delete Paragraph command in the Edit menu on the toolbar, or just type Ctrl-Del. You will be using Ctrl-Del a lot to get rid of unwanted junk in execution groups. Be careful, though; it will kill *the whole paragraph* . So unless you are sure of what is in the paragraph in question, click on the paragraph symbol and see how the paragraphs are arranged. Note that you can also use the mouse to highlight text to be deleted by clicking and holding at the beginning of the offensive material and dragging the cursor to highlight everything that has to go. Then just press Del. Ok, so use Ctrl-Del, or mouse highlighting and Del to get rid of the error message. Then use the mouse to move the cursor to the end of 1+1 , tack on the semicolon, hit Enter, and you will have used Maple to find out that 1+1=2. (Note that the answer comes out in blue in the center of the page). And all of the cut, paste, copy, etc., stuff you are used to is available in Maple under the Edit icon on the toolbar. One final note. You probably noticed the extra execution group after 1+1;. This extra one is there as a trap. Whenever Maple performs the commands in one execution group, it skips down to the next one. If the next one is way far down in some text (like here in this section) your desired result will be clear off the screen and you will need to move back up there to see it. You will see lots of traps like this in this document, but in Maple code that is not heavily documented, they aren't necessary.

Maple will allow you to do more than one calculation inside an execution group. You can do this in two ways: (i) with several commands on a single line separated by semicolons, or (ii) several commands stacked vertically, each with its own >. To get extra >'s, just move the cursor to the left of the first > and hit Enter. Examples that you can execute are displayed below; try them out. Oh, and notice how the # symbol can be used to add brief comment lines in red right on the command line. The rule is that anything after # is a comment

> **1+1;2+2;3+3; # three additions in a row**

4

6

> **1+1;**> **2+2;**> **3+3;**

2

4

6

OK, hopefully this is enough to get you around on the window. There are a lot of things that won't be mentioned here, but they are all described in online help and you can find most of them on the toolbar. This might be a good time to take a tour of the toolbar to see what's up there. To begin, click on Help and enable Balloon Help; then move the cursor across each icon to see what each one is about. And for a more thorough introduction to the way the window is managed, click on Help and try the New Users Tour. After you click on the New Users Tour go to the bottom of the introduction page and click on the blue hyperlink called **Click here to begin the New Users's Tour** , then select item (2), The Worksheet Environment . Oh, and when you want to close a Help screen, use Ctrl-F4 or the X at the upper right-hand corner of the window (in Windows).

>

Problem 1.1

Make an execution group and document it before the > sign with the phrase:

Add the integers from 1 to 10.

Then build a Maple command line to get the answer to $1+2+3+\dots+10$.

[Go to top of section](#)

Quick guide to managing the worksheet

[Debugging](#)

This is just a collection of recipes for making the worksheet look like you want it to. I have given you several control sequences here; you can also do these things from the Edit and Insert menus.

Make an execution group:

Ctrl-j makes one after the execution group or paragraph the cursor is in.

Ctrl-k makes one before the execution group paragraph the cursor is in.

Adding another > to an execution group

Use the arrow keys, or the mouse, to move the cursor just to the left of a > already in the group. Hit Enter and another one will appear just above.

Make a paragraph for text inside an execution group

Shift-Ctrl-j makes one after the cursor.

Shift-Ctrl-k makes one before the cursor (you also get one if you left arrow past the < sign and hit Enter)

Slip a paragraph for text between two execution groups

Click on the big [at the left of the group; it will become a double line; then type **Shift-Ctrl-k** to get one before, **Shift-Ctrl-j** to get one after.

Delete a paragraph or execution group

Put the cursor in the paragraph or execution group to be deleted and type **Ctrl-Del**. To kill everything in the execution group you may have to do this several times.

Enter equations or expressions within text

Click on Σ on the toolbar (or type **Ctrl-r**) then watch the equation being typed in the window on the toolbar. The syntax for entering typeset mathematics is Maple's own syntax. For instance, to display the

integral of the sine function from 0 to 1 you would type this: `int(sin(x),x=0..1)` but with **Ctrl-r** turned on,

like this $\int_0^1 \sin(x) dx$. (Highlight this integral expression with the mouse and watch it appear as

Maple code on the toolbar.) Hit Enter when it is complete, then click on T on the toolbar (or type **Ctrl-t**) to go back to text mode. On some systems you will need to click on the check mark two buttons to the left of the math insertion window on the tool bar, followed by **Ctrl-t** (or clicking on T).

Split or join execution groups

See the Edit menu on the toolbar.

Remove all of the blue Maple output from the worksheet

See the Edit menu.

Close all of the sections and subsections in a worksheet

See the bottom of the View menu on the toolbar.

Removing the output and closing all of the sections fully restores this book to its original condition.

Other interesting tricks

Explore the toolbar and see what's up there. Of special interest are:

- (a) Tile in the Window menu allows you to see the worksheet and do your assignment in side-by-side windows. Cascade under Window overlays them.
- (b) You can get bold, italic, and underlined text with **B** , *I*, and u on the toolbar.
- (c) Use the magnifying glass icons on the toolbar to change the size of the type.
- (d) Notice the curvy arrow icons on the toolbar--they undo the last operation and redo the last undo.
- (e) The stop sign will kill a calculation that has gone on so long you can't stand it anymore.

Problem 1.2

Open a new worksheet. At the top of the worksheet, above the first execution group, open up some white space and put in a bold title that says Problem 1.2. Below this title, in italics, put your name. Then expand the execution group so that it contains four >'s instead of just one and put comment lines that say "Comment 1", "Comment 2", etc., above each >.

[Go to top of section](#)

Online help

You probably already know how to use online help in programs, so when you click on the Help icon at the top of the page you will be in familiar territory. But there is a quicker way to get help directly at the Maple prompt: just use the question mark. For example, suppose you are pretty sure that Maple knows how to find the prime factors of big numbers, but you don't know the command. Try this.

```
> ?factor;
```

When you do you will find that this is the wrong command because it does algebra on polynomials. But if you read a bit, and look at the helpful See Also links at the bottom of the page you will see that the command you want is **ifactor**, so you either go there or use

```
> ?ifactor;
```

Now help will tell you that this is just what you want, and you can try

```
> ifactor(5040);
```

$$(2)^4 (3)^2 (5) (7)$$

You can find lots of interesting stuff in Maple this way; happy hunting. And remember that the quick way to kill the help windows is with **Ctrl-F4**.

```
>
```

Problem 1.3

Use online help to find the Maple commands for integrating a function and for taking a derivative of a function.

Problem 1.4

Use online help to find the Maple command for taking the second derivative of a function.

[Go to top of section](#)

Special constants

[Debugging](#)

There are several constants pre-loaded in Maple that are useful for physics. They are:

> **Pi;I;infinity;**

π

I

∞

Pi is the number Pi to about as many significant figures as you want. To see its digits, use the Maple command **evalf** to convert a number to floating point form

> **evalf(Pi,100);**

3.141592653589793238462643383279502884197169399375\
10582097494459230781640628620899862803482534211706\
8

and you should be impressed. I is the imaginary unit, i.e., $I = \sqrt{-1}$; try executing

> **sqrt(-5);**

$I\sqrt{5}$

Infinity is, of course, not a number, but an idea, and only makes sense in limits. And Maple does the right thing in calculations involving limits, like

```
> limit(exp(-x),x=infinity);
```

0

or

```
> sum('1/2^n','n'=1..infinity);
```

1

```
>
```

We will do more of this kind of thing in Chapter 4.

Problem 1.5

Compare the Maple commands `cos(pi)` and `cos(Pi)` by trying them in an execution group. Explain why they don't behave the same way. (This issue will be discussed in more detail in the section on variable names).

Problem 1.6

Calculate the complex division problem $(3+4i)/(7-6i)$ in a Maple execution group.

Problem 1.7

Calculate the limit, as x approaches infinity, of the expression $(3-5x)/(7+3x)$ using a Maple execution group.

[Go to top of section](#)

Variable names

[Debugging](#)

Like all other modern computer languages, Maple allows you to name a variable just about anything you want, with some exceptions. In C and Fortran, for instance, you can't use "sin" as a variable name because this name is reserved for the sine function. You can probably think of other obvious reserved names like this ("cos", "tan", "log", etc.), but there are only about a hundred of these and you have the whole alphabet at your disposal, so it's not a big problem. In Maple you have to be more careful because there are about 3000 commands whose names are reserved for system use. But Maple will tell you if you have made a bad choice, and the possibilities for naming variables are still nearly endless.

Now let me warn you about a bad habit into which this naming freedom seems to have led your generation of computer users. Here is how some students I have worked with would write Newton's law of universal gravitation in a computer code:

```
Newton's_Gravitational_Constant*Mass_1*Mass_2/Distance_Between_The_Masses^2
```

Here is what this would look like in a Maple execution group:

```
> Newton's_Gravitational_Constant*Mass_1*Mass_2/Distance_Between_The_Masses^2;
```

$$\frac{\text{Newton's_Gravitational_Constant } \text{Mass}_1 \text{ Mass}_2}{\text{Distance_Between_The_Masses}^2}$$

Now look; this form is certainly well documented, but you are going to spend a great deal of your life in programming typing code and reading code that has already been typed. **Avoid clutter like the flu!** Here's how I would suggest you write it in Maple:

```
> G*m1*m2/r12^2;
```

$$\frac{G m_1 m_2}{r_{12}^2}$$

```
>
```

The prosecution rests. I don't know who is responsible for the current popularity of the underscore character and capitalization in variable names, but if I were you I would use this style very sparingly, if

at all. When I program I choose variable names to be as much like the symbols used in physics textbooks and research papers as possible. These would be things like a, m, F, c, q, L, h, etc..

But Maple has the ability to go even further toward the standard physics style because it has Greek letters! To get them just write out their names in English, as in the example below.

> **alpha*beta/delta;**

$$\frac{\alpha \beta}{\delta}$$

The capital Greek letters are also available, and even though some of them look like English capital letters, they are not the same, as you can see in the example below:

> **Alpha*A;Beta+B;Chi/Gamma;**

$$A A$$

$$B + B$$

$$\frac{\text{Chi}}{\Gamma}$$

Well, OK; not all of the Greek capital letters are produced this way. Some of them you do this way

> **CHI;**

$$X$$

You can get all of them by creatively switching between a capital letter first and all capitals. If you persevere you can work in Maple with beautifully typeset equations just like you see in your textbooks. (And if you don't want to memorize their spellings and capitalizations, click on View, Palettes, Symbol Palettes and you will have a table of symbols you can get with the mouse.)

Note that a few are not available for your use:

> **gamma:=4;Chi:=3;**

```
Error, attempting to assign to `gamma` which is protected
```

```
Error, attempting to assign to `Chi` which is protected
```

```
>
```

These two, and possibly others, are used by the system. It may be possible to *unassign* them so that they are available for your own use, but I recommend against it. The more you work strictly within the Maple system the better off you will be; or in the language of Sunday afternoon in America, "take what the defense gives you."

Also avoid putting subscripts on variables, even though Maple's typesetting ability will let you do it. For instance you might want to work with the variables x_1 and x_2 which you get by doing this in Maple

```
> x[1]:=1;x[2]:=2;
```

```
x1 := 1
```

```
x2 := 2
```

They look great, but now watch what happens if you give x a value, then display x_1

```
> x:=3;
```

```
x := 3
```

```
> x[1];
```

```
31
```

I think we can all agree that this is not going to help us do mathematics. So if you want to code x_1 in Maple, call it x1. And as you choose variables, experiment a little with the names and try for elegance and simplicity.

>

Problem 1.8

Set the variable θ to have the value $5/3$ and then coax the Maple command $\cos(\theta)$ into giving you $\cos(5/3)$ as a floating point number.

[Go to top of section](#)

What can a variable represent?Debugging

In standard computing languages like Fortran, C, and Matlab variables can only represent a limited number of different things, like integers, floating point numbers, complex numbers, and character strings, to name nearly all of them. But in Maple there are over 100 different things a variable can represent. This makes life much more complicated. For instance, in Fortran or Matlab the statement

a=5

would assign to the variable a the value 5. But in Maple the statement

> **a=5;**

$$a = 5$$

doesn't do assignment at all. Instead it is just the really stupid equation $a = 5$. Maple is capable of solving this equation and discovering that a is 5, but it is not a statement that assigns a value. In order that Maple be able to work directly with equations (expressions containing the symbol =), the assignment statement has been given a different form. To assign to variable a the value 5 the Maple statement would be

> **a:=5;**

$$a := 5$$

One of the clearest ways of indicating the difference is to note that the following statement in Maple makes sense:

> **Eq:=b=5;**

$$Eq := b = 5$$

This statement means that the variable Eq contains the equation $b = 5$. Here's a less dumb example:

> **Eq2:= x^2 + y^2 = R^2;**

$$Eq2 := 9 + y^2 = R^2$$

Well, this probably looks a little strange; how come that 9 is there instead of an x? Well, maybe it isn't, depending on how you got here, but if you get a 9, here's why. In the last section on variable names x was set to 3, and Maple remembers forever. This will bite you over and over. The quickest way to cure it if you don't care about the stuff you have done before is to just restart Maple, like this:

> **restart;**

Now use the mouse to move the cursor back up to the Eq2 assignment, click on the red text, and it will execute properly. Note that this works even though the **restart** is below the Eq2:=... . Maple is not like C or Matlab. Order in the worksheet is not what's important--order in time spent at the keyboard is what matters.

So, what are the possible things that a variable can represent? The subsections given below are a partial list of the ones you are most likely to encounter. They are just Maple examples without comment. We will discuss each one in detail sometime during the course. To get a full look at what's possible, use

> **?type**

And if you are confused about the type of, say, a variable named a, you can use the Maple command **whattype(a)**.

Problem 1.9

On a clean worksheet use the Maple command **b:=a=5** and then use the type command to check the type

of b by using the commands **type(b,integer)**, **type(b,float)**, and **type(b,equation)** . Explain why the results of these three commands make sense to Maple.

>

The sections below contain examples of the most important data types we will be using in this book.

Complex numbers

Note that $I = \sqrt{-1}$.

> **cn1:=(1+3*I)*(5-2*I);**

cn1 := 11 + 13 I

Equations

> **Eq1:=sin(x)=x;**

Eq1 := sin(x) = x

Floating point numbers

Maple usually keeps 10 significant figures in floating point (real number) calculations. But you can increase it by changing the value of the Maple constant **Digits** , like this:

> **f1:=4./3.;**

f1 := 1.333333333

> **Digits:=50;**

Digits := 50

> **f1:=4./3.;**

$$Fn2 := x \rightarrow \frac{\cosh(x^3)}{1+x^2}$$

This does behave like a function because you give it an argument and it gives you an answer back

> **Fn2(2.);**

298.0958322

Or you can define a function with multiple arguments like this

> **Fn3:=(x,y,z)->x*y^2*z^3;**

$$Fn3 := (x, y, z) \rightarrow x y^2 z^3$$

> **Fn3(1,2,3);**

108

Hence, the **->** function notation makes functions like you are used to from math classes. So why would you even want to mess with expressions? Because Maple often has an easier time doing algebra with them, integrating them, differentiating them, plotting them, and using them to build differential equations. So you have to get used to both. I try to remember it this way: a Maple expression looks like a function in a math book; a Maple function is a machine that takes input and gives back output.

>

Integers

> **I1:=12345678987654321;**

I1 := 12345678987654321

Matrices

> **M1:=matrix([[1,2],[3,4]]);**

$$M1 := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Polynomials

> **P1:=y^3+y^2+y-1;**

$$P1 := y^3 + y^2 + y - 1$$

Strings

> **S1:="Mary had a little lamb.";**

S1 := "Mary had a little lamb."

Vectors

> **V1:=Vector([1,2,3]);**

$$V1 := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

>

Lists and sets

A list is a sequence of numbers, expressions, or any other legal data type enclosed in square brackets. They can be loaded either by hand or by means of the **seq** command.

> **s1:=[1,2,4,6,8,12];**

s1 := [1, 2, 4, 6, 8, 12]

> **s2:=[seq(2*n^2-3*n+2,n=1..6)];**

$$s2 := [1, 4, 11, 22, 37, 56]$$

Their elements can be referred to with Maple's square bracket subscript notation

```
> s1[3];s2[2..5];
```

4

$$[4, 11, 22, 37]$$

A set is similar to a list, but not quite. It is enclosed in curly brackets and Maple, instead of you, decides what order the elements should be listed in. The most important difference is that a list may contain repeated elements, but a set may not. Sets are often used in the arguments of Maple commands. In the example below note that the list stays the way it was defined while the set is re-ordered and reduced to unique elements.

```
> s1:=[1,3,2,6,4,3,2];
```

$$s1 := [1, 3, 2, 6, 4, 3, 2]$$

```
> s2:={1,3,2,6,4,3,2};
```

$$s2 := \{1, 2, 3, 4, 6\}$$

>

[Go to top of section](#)

Getting the answer

[Debugging](#)

Now we have come to the second most annoying thing about Maple: sometimes it doesn't feel like giving you the answer to your problem. Here are some examples.

Suppose you want to know the sine of the number 1 (in radians; all calculus is done in radians, so forget about degrees). You might think this would work:

```
> restart;
```

```
> sin(1);
```

```
sin(1)
```

This is not what we had in mind; we already know we want $\sin(1)$, but what is the answer? Maple is having trouble here because it doesn't think it makes sense to take the sine of an integer constant; it expects floating point numbers. See what happens with the small addition of a decimal point

```
> sin(1.);
```

```
.8414709848
```

Difficulties like this are not always so easy to fix, so Maple provides a command to take care of it: `evalf`. It would be used in this example this way:

```
> sin(1);
```

```
sin(1)
```

```
> evalf(%);
```

```
.8414709848
```

```
>
```

The `%` sign means "the last result Maple calculated" (like Ans on some calculators). It is $\sin(1)$ in this case, and `evalf` stands for "evaluate as a floating point number". You will be using `evalf` a lot because Maple will be doing its symbolic thing when you want a numeric result. It's just the price you have to pay to have a calculator that is at home in both the symbolic and numerical worlds. **Warning: if you get in the habit of using % routinely, you will routinely make mistakes. Only use it sparingly. Most of the time you should assign results to variable names, then work with these names to avoid confusion, like this:**

```
> s:=sin(1);
```

```
s := sin(1)
```

> **evalf(s);**

.8414709848

Oh; and what's the number one most annoying thing about Maple? It doesn't give you very good error messages when you louse up. It tends either to just parrot back to you the thing you asked it to do, or it gives you gobbledygook like "Can't evaluate Boolean". But be patient; it is a great tool and it is improving all the time. It is definitely worth learning.

>

[Go to top of section](#)

Debugging (when bad things happen to good programmers)

[Debugging](#)

Everybody who writes or uses software has something in common--their programs don't work. And when this happens the most important skill you can have is the ability to find out exactly what went wrong. Here are some beginning tips for debugging Maple. More information is in the Debugging section at the end of the worksheet.

evalf (look at the numbers)

Suppose you wanted to know what the function $\frac{\sin(a^2 x^2)}{\sin(ax)^2}$ looks like. Maple's plot command is a

wonderful way to see what functions look like, so you could define the function this way (the **restart** command is to make sure that this example is starting out with a clean slate)

> **restart: f:=x-> sin(a^2*x^2)/sin(a*x)^2;**

$$f := x \rightarrow \frac{\sin(a^2 x^2)}{\sin(ax)^2}$$

then use

> **plot(f(x),x=-1..1);**

Plotting error, empty plot

You will come to hate this error message. What it really means is that there is something wrong with the function $f(x)$, but because Maple is so powerful and allows you to do all kinds of things, it can't tell you what the problem is. Hopefully at some point in time Maple will have better error messages. But you have a problem right now. When this happens, **evalf** can help. Let's check a particular value of this function to see if we can get a hint about what's wrong

> **evalf(f(1.));**

$$\frac{\sin(1. \alpha^2)}{\sin(2. \alpha)^2}$$

This is an error message in disguise. What it means is that Maple tried to give you a number back, but couldn't because you hadn't told it what α was. As soon as you give α a value, the plot will work. (Set a to be 5 in the execution group below and go back up and execute the plot.)

>

Here's another example. Suppose you forgot the value of $\sin\left(\frac{\pi}{2}\right)$ and needed Maple to tell you, so

you try

> **s:=sin(pi/2);**

$$s := \sin\left(\frac{1}{2} \pi\right)$$

"No!", you scream, "GIVE ME THE NUMBER!" The Maple command for this would be

> **evalf(s);**

$$\sin(.5000000000 \pi)$$

This, again, is an error message. It says that Maple tried to give you the number, but couldn't because it didn't know what π is. I have led you into error by using Maple's ability to deal with Greek letters.

The symbol pi is just the Greek letter π ; what you want is the ratio of circle area to the square of circle radius, which is Pi in Maple. When you do it this way it works

```
> sin(Pi/2.);
```

1

```
>
```

When Maple gives you back something unexpected, try to turn it into a number that you can check by using **evalf**. As you try to get a number out you will probably discover what the problem is.

[Go to top of section](#)

check := vs =

You will make this mistake 100 times at least: only use **=** when you are defining an equation; assignment, i.e., you want a to be 5, is done with **a:=5**.

plot (make a picture)

Sometimes you will need to look at a lot more than just one number to see what went wrong, and the best way to look at a thousand numbers is to make a picture. For instance, suppose you tried to do the integral

```
> restart;
```

```
> int(sin(x)/(1-cos(x)),x=0..Pi);
```

∞

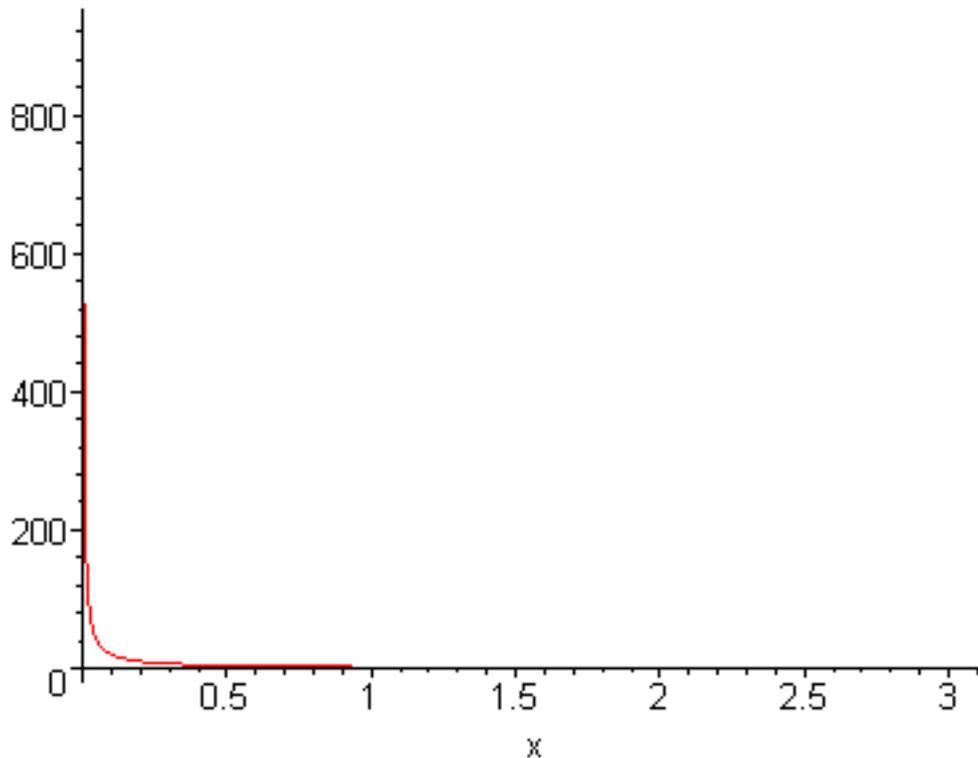
You thought you were going to get a number, but Maple gave you infinity. You could use floating point arguments to check the function like this

```
> sin(1.)/(1-cos(1.));
```

1.830487722

but that looks fine. To check a whole bunch of values over the integration range, make a plot using Maple's command `plot`:

```
> plot(sin(x)/(1-cos(x)),x=0..Pi);
```



```
>
```

Now you can see that the integrand is singular at $x=0$, so Maple was right: the answer is infinity. Making plots is a great way to debug.

assume (tie Maple down)

Sometimes Maple can't do what you want it to do because you are thinking about a specific problem and Maple is thinking generally. For instance, suppose you wanted to do the integral

```
> restart;
```

```
> int(exp(-a^2*x^2),x=0..infinity);
```

Definite integration: Can't determine if the integral is convergent.

Need to know the sign of --> a^2

Will now try indefinite integration and then take limits.

$$\lim_{x \rightarrow \infty} \frac{1}{2} \frac{\sqrt{\pi} \operatorname{erf}(a x)}{a}$$

You wanted a formula containing a , but instead you got this mess. This time Maple gives you a helpful error message--since Maple can handle complex numbers, it can't tell whether a^2 is positive or negative. It will give you the answer if you give it a little help with **assume**

> **assume(a,real);**

> **int(exp(-a^2*x^2),x=0..infinity);**

$$\frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{a^2}}$$

>

Look the **assume** command up in online help; it will help you keep Maple under control.

Warning: Once you have used **assume** on a variable, however, you may have trouble having Maple give you numbers. The trouble happens when you assume something about a, use a in an expression and assign it to another variable (say b), then give a a value, and finally try to evaluate b. Observe:

> **restart;**

> **assume(a,real);**

> **b:=cos(a);**

$$b := \cos(a)$$

> **a:=2.;**

$$a := 2.$$

> **evalf(b);**

$$\cos(a)$$

Now watch what happens to the same sequence without the assume

> **restart;**

> **b:=cos(a);**

$$b := \cos(a)$$

> **a:=2.;**

$$a := 2.$$

> **evalf(b);**

$$-.4161468365$$

The lesson here is only to use **assume** when absolutely necessary.

Finally, you may need to assume two conditions on a variable, like both $c > 0$ and $c < 1$. Maple lets you do this in two ways. You can either put both conditions inside a single **assume** command like this

> **assume(c>0,c<1);**

or you can do it in two (or more) steps using the command **additionally**, like this

> **assume(c>0);**

> **additionally(c<1);**

The reason you can't just use two **assume** commands is that whenever you use **assume** on a variable it erases all previous assumptions set on that variable.

>

restart (punt)

Sometimes Maple has a hard time giving you what you want because it has become so cluttered with everything you have been doing in a session that things you have forgotten, but that it remembers, interfere with what you are trying to do. I can only give you a simple example here, but this kind of thing happens all the time and it is quite often easier to just start over than to try to figure out what went wrong. Suppose you were exploring the Bessel function $J_0(x)$ by checking a few values like this

> **restart;**> **x:=1.5;BesselJ(0,x);** $x := 1.5$ $.5118276717$

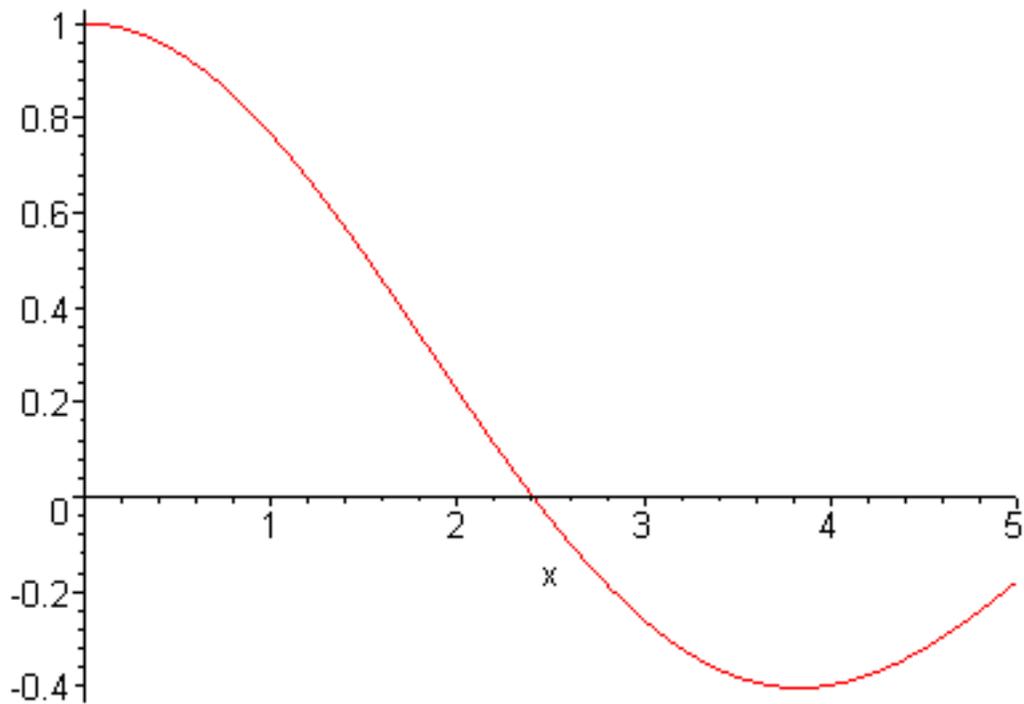
And then after trying a few different values of x , you decided to plot the function like this

> **plot(BesselJ(0,x),x=0..5);**

Error, (in plot) invalid arguments

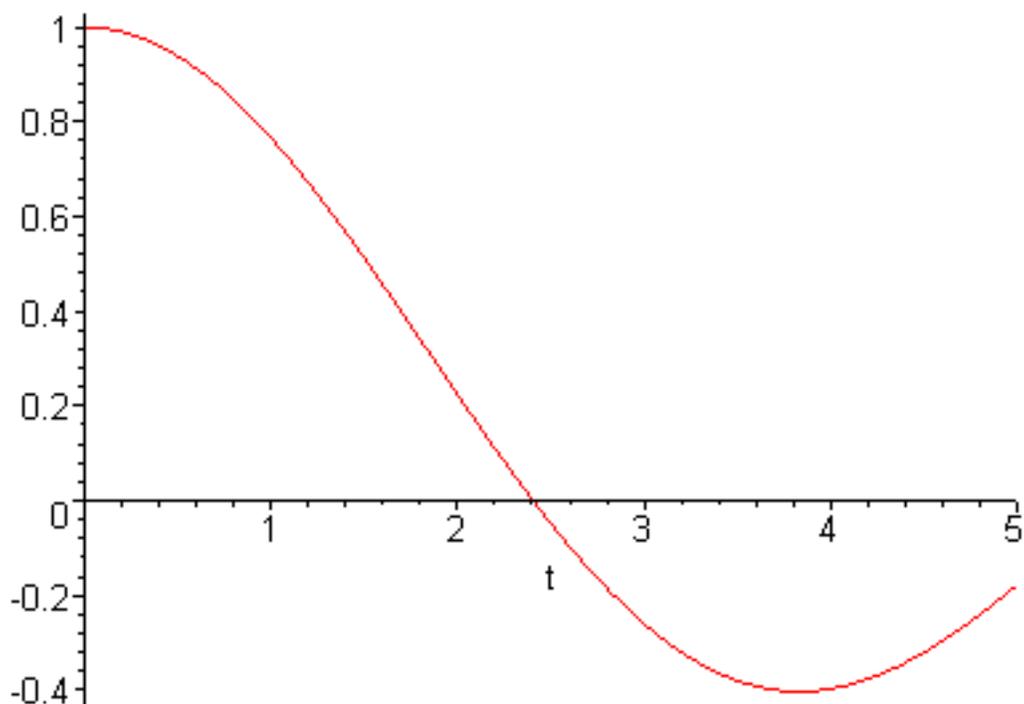
This is a pretty mysterious error message, and you have to know quite a bit about Maple to be able to tell what's wrong. But if you do this

> **restart: plot(BesselJ(0,x),x=0..5);**



all is well. The **restart** command is the Maple equivalent of rebooting, and you will be using it regularly. (Note: the plot failed because we tried to plot with the variable `x`, which had been given a value of 1.5 just before the plot. Using an unassigned variable name works just fine:)

```
> plot(BesselJ(0,t),t=0..5);
```



>

[Go to top of section](#)

Cool stuff you can do right now

Even knowing the little bit you have learned right now you can do some rather amazing things. Open each subsection and see what you can do.

Integration

[Debugging](#)

Integration is done with the `int` command.

Here's an indefinite integral $\int e^{(-x)} \sin(x) dx$:

> **restart;**

> **int(exp(-x)*sin(x),x);**

$$-\frac{1}{2} e^{(-x)} \cos(x) - \frac{1}{2} e^{(-x)} \sin(x)$$

Here's a definite integral $\int_0^1 e^{(-x)} \sin(x) dx$

> **s1:=int(exp(-x)*sin(x),x=0..1);**

$$s1 := -\frac{1}{2} e^{(-1)} \cos(1) - \frac{1}{2} e^{(-1)} \sin(1) + \frac{1}{2}$$

This is a perfectly valid expression, but perhaps you want a number?

> **evalf(s1);**

.2458370070

Here's an integral Maple can't do

> **s2:=int(exp(-x^4)*cos(x),x=0..1);**

$$s2 := \int_0^1 e^{-x^4} \cos(x) dx$$

so it just prints it nicely and lets you worry about how to get an actual number out. You do this by using **evalf** to ask Maple to do it numerically

> **evalf(s2);**

.7363519309

>

Problem 1.10

Find the numerical answer to $\int_0^2 e^{\sqrt{x}} dx$. Put a comment on the same line as the command with

Maple's # notation.

[Go to top of section](#)

Differentiation

Debugging

Take the derivative of any function using **diff**

> **restart;**

> **f:=sin(x)/sqrt(1+x^2);**

$$f := \frac{\sin(x)}{\sqrt{1+x^2}}$$

> **d1:=diff(f,x);**

$$d1 := \frac{\cos(x)}{\sqrt{1+x^2}} - \frac{\sin(x) x}{(1+x^2)^{\frac{3}{2}}}$$

Perhaps you don't like the look of this and would rather have a common denominator. This is called normal form and can be obtained this way (we will do more of this in Chapter 9)

> **normal(d1);**

$$\frac{\cos(x) + \cos(x) x^2 - \sin(x) x}{(1+x^2)^{\frac{3}{2}}}$$

You can do second derivatives too

> **d2:=diff(exp(-x)*tan(x),x\$2);**

$$d2 := e^{(-x)} \tan(x) - 2 e^{(-x)} (1 + \tan(x)^2)$$

$$+ 2 e^{(-x)} \tan(x) (1 + \tan(x)^2)$$

Maybe you think this could look prettier? Try

> **simplify(d2);**

$$3 e^{(-x)} \tan(x) - 2 e^{(-x)} - 2 e^{(-x)} \tan(x)^2 + 2 e^{(-x)} \tan(x)^3$$

then try to pull out the common exponential factor with

> **factor(%);**

$$e^{(-x)} (3 \tan(x) - 2 - 2 \tan(x)^2 + 2 \tan(x)^3)$$

and then get the polynomial in $\tan(x)$ to be in the right order

> **sort(%);**

$$(2 \tan(x)^3 - 2 \tan(x)^2 + 3 \tan(x) - 2) e^{(-x)}$$

And now that I like the final answer I will assign it to d2. This is one of the few times that I use the previous answer variable `%`. It is useful here because if one of the algebra commands I use to get an improved form doesn't work, I haven't messed up what's in d2, so I can just keep trying things until I get what I get.

> **d2:=%;**

$$d2 := (2 \tan(x)^3 - 2 \tan(x)^2 + 3 \tan(x) - 2) e^{(-x)}$$

These algebra commands, and more, are discussed in Chapter 9. During the semester you should get used to going to Chapter 9 and looking around whenever you want Maple put an expression in a form you like better.

>

Problem 1.11

Take the first derivative of the function $\cos(\cos(\cos(\cos(x))))$. Put a comment about what you are doing in text within the execution group and above the **diff** command.

[Go to top of section](#)

Build a sequenceDebugging

In calculus you studied sequences of numbers that tend to a limit and infinite sums of such sequences. In Maple sequences are built with the **seq** command. It works this way

> **restart;**

> **s:=seq((5+n)/(3+n),n=1..50);**

$$s := \frac{3}{2}, \frac{7}{5}, \frac{4}{3}, \frac{9}{7}, \frac{5}{4}, \frac{11}{9}, \frac{6}{5}, \frac{13}{11}, \frac{7}{6}, \frac{15}{13}, \frac{8}{7}, \frac{17}{15}, \frac{9}{8}, \frac{19}{17}, \frac{10}{9}, \frac{21}{19}, \frac{11}{10}, \frac{23}{21}, \frac{12}{11},$$

$$\frac{25}{23}, \frac{13}{12}, \frac{27}{25}, \frac{14}{13}, \frac{29}{27}, \frac{15}{14}, \frac{31}{29}, \frac{16}{15}, \frac{33}{31}, \frac{17}{16}, \frac{35}{33}, \frac{18}{17}, \frac{37}{35}, \frac{19}{18}, \frac{39}{37}, \frac{20}{19}, \frac{41}{39},$$

$$\frac{21}{20}, \frac{43}{41}, \frac{22}{21}, \frac{45}{43}, \frac{23}{22}, \frac{47}{45}, \frac{24}{23}, \frac{49}{47}, \frac{25}{24}, \frac{51}{49}, \frac{26}{25}, \frac{53}{51}, \frac{27}{26}, \frac{55}{53}$$

or

> **evalf(s);**

1.500000000, 1.400000000, 1.333333333, 1.285714286,
 1.250000000, 1.222222222, 1.200000000, 1.181818182,
 1.166666667, 1.153846154, 1.142857143, 1.133333333,
 1.125000000, 1.117647059, 1.111111111, 1.105263158,

1.100000000, 1.095238095, 1.090909091, 1.086956522,
 1.083333333, 1.080000000, 1.076923077, 1.074074074,
 1.071428571, 1.068965517, 1.066666667, 1.064516129,
 1.062500000, 1.060606061, 1.058823529, 1.057142857,
 1.055555556, 1.054054054, 1.052631579, 1.051282051,
 1.050000000, 1.048780488, 1.047619048, 1.046511628,
 1.045454545, 1.044444444, 1.043478261, 1.042553191,
 1.041666667, 1.040816327, 1.040000000, 1.039215686,
 1.038461538, 1.037735849

which gives us a hint that the sequence is approaching a limit of 1.

Or you can build a sequence of expressions, like the terms of a power series. Here are the terms in the series for the exponential function

> **seq(x^n/n!,n=1..20);**

$x, \frac{1}{2}x^2, \frac{1}{6}x^3, \frac{1}{24}x^4, \frac{1}{120}x^5, \frac{1}{720}x^6, \frac{1}{5040}x^7, \frac{1}{40320}x^8,$
 $\frac{1}{362880}x^9, \frac{1}{3628800}x^{10}, \frac{1}{39916800}x^{11}, \frac{1}{479001600}x^{12},$
 $\frac{1}{6227020800}x^{13}, \frac{1}{87178291200}x^{14}, \frac{1}{1307674368000}x^{15},$
 $\frac{1}{20922789888000}x^{16}, \frac{1}{355687428096000}x^{17},$
 $\frac{1}{6402373705728000}x^{18}, \frac{1}{121645100408832000}x^{19},$
 $\frac{1}{2432902008176640000}x^{20}$

>

Problem 1.12

Generate the first 15 terms of the sequence defined by $s_n = \text{diff}\left(\frac{\cos(x)}{1+x}, x \text{ \$ } n\right)$. Put a comment about what you are doing in text below the seq command, but within the same execution group.

Sum a series

Debugging

Find a formula for the sum of the integers raised to the 6th power using **sum**

> **restart;**

> **s1:=sum(k^6,k=1..n);**

s1 :=

$$\frac{1}{7}(n+1)^7 - \frac{1}{2}(n+1)^6 + \frac{1}{2}(n+1)^5 - \frac{1}{6}(n+1)^3 + \frac{1}{42}n + \frac{1}{42}$$

Find the result of summing the reciprocals of the integers raised to the 6th power from 1 to ∞

> **s2:=sum(1/n^6,n=1..infinity);**

$$s2 := \frac{1}{945} \pi^6$$

then convert this to a numerical value

> **evalf(s2);**

1.017343063

>

Problem 1.13

Find the sum of the reciprocals of the squares of the odd integers from 1 to ∞ . Put a comment about what you are doing in a separate paragraph above and outside the execution group containing the **sum** command.

Taylor series

Debugging

One of the most useful tools used by physicists to get approximate answers to difficult problems is the Taylor series. You have probably forgotten this result, so here it is: any function $f(x)$ which is infinitely differentiable can be approximated in the neighborhood of $x = a$ by a polynomial in $(x - a)$ which is given by Taylor's famous formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

This is very useful in getting approximations. For instance, the quantity $\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ appears all

the time in relativity theory. It is sometimes nice to have an approximate form for this formula if $\frac{v}{c}$ is small. Use the **taylor** command to find it.

> **restart;**

> **taylor(1/sqrt(1-x^2),x,5);**

$$1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + O(x^6)$$

>

where the 5 means get terms up through x^5 in the expansion. The term $O(x^6)$ in the answer

means that the error in this expansion is of order x^6

•

Problem 1.14

Find an approximate formula for the function $e^{\cos(x)}$ including terms up through x^{10} . Display

the coefficients of the series in both rational and floating point form. Put a comment about what you are doing in a text paragraph outside and below the execution group containing the **taylor** command.

[Go to top of section](#)

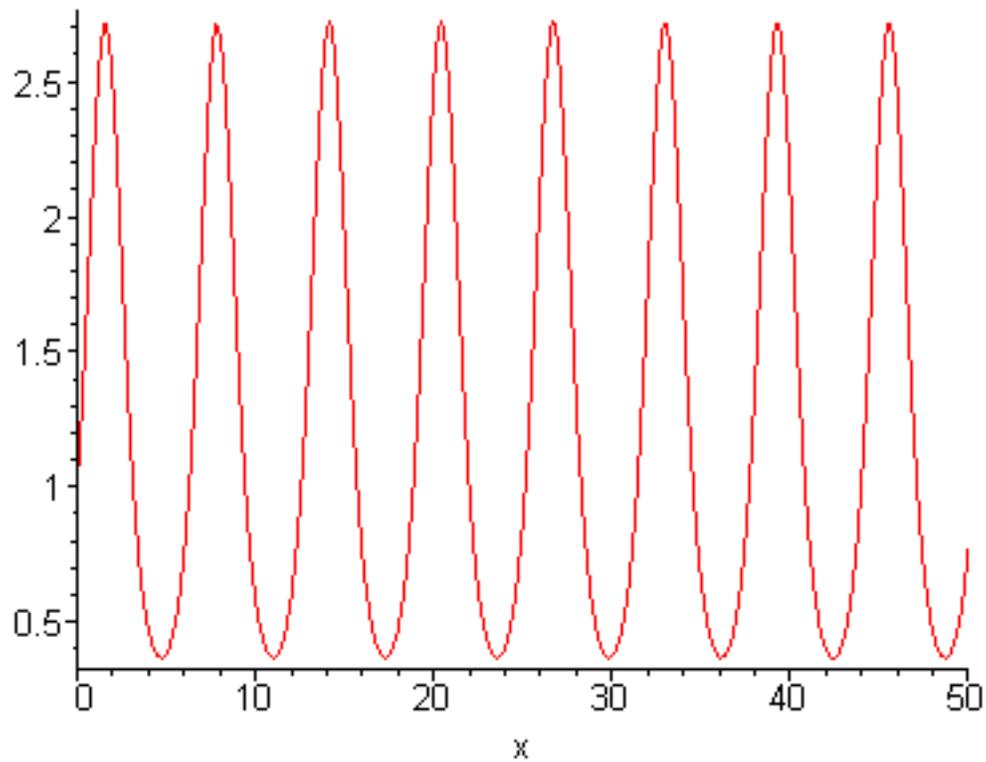
Plot functions

[Debugging](#)

If you want to quickly find out what some function that you have run across looks like, use **plot**

> **restart;**

> **plot(exp(sin(x)),x=0..50);**



>

Problem 1.15

Make a plot of the function $e^{\left(-\frac{x}{25}\right)} \cos(x)$ from $x = 0$ to $x = 100$. Then put a comment

about what you are doing in a paragraph between the two execution groups.

Solve hard equations

[Debugging](#)

Suppose you need a solution of $e^{(-x)} = x$. Use `solve` on the equation

> **restart;**

> **sol:=solve(exp(-x)=x,x);**

$sol := LambertW(1)$

Well, this is an answer in terms of some function you have never heard of when what you really wanted was just a number. No problem; just use

> **evalf(sol);**

.5671432904

Or if you already know that there isn't going to be a symbolic solution and you just want a number, you could tell Maple to just numerically solve this equation with Maple's numerical equation solver **fsolve**, telling Maple to start looking for a solution near $x=1$:

> **fsolve(exp(-x)=x,x=1);**

.5671432904

If you don't give it a good guess it may take longer to get the answer or it may get lost and go to infinity instead of finding the answer.

Problem 1.16

Find the solution of the equation $\cos(x) = x$. Put a comment about what you are doing in a paragraph within the execution group just above the command. Within this paragraph of text use Maple's math mode (Σ on the toolbar) to include the equation $\cos(x) = x$.

Maple will also solve equations (even systems of equations) for you symbolically. For example, here is how Maple solves the quadratic equation symbolically:

> **Q:=a*x^2+b*x+c=0;**

$Q := ax^2 + bx + c = 0$

> **solve(Q,x);**

$$\frac{1}{2} \frac{-b + \sqrt{b^2 - 4ac}}{a}, \frac{1}{2} \frac{-b - \sqrt{b^2 - 4ac}}{a}$$

And here's the symbolic solution of the general problem of two equations in two unknowns, with x and y unknown and $a-f$ as the coefficients:

> **E1:=a*x+b*y=e;E2:=c*x+d*y=f;**

$$E1 := a x + b y = e$$

$$E2 := c x + d y = f$$

> **solve({E1,E2},{x,y});**

$$\left(y = -\frac{af - ce}{-ad + bc}, x = \frac{bf - ed}{-ad + bc} \right)$$

You may find this an easier way of handling simultaneous equations than using Maple's linear algebra package.

>

Solve differential equations

Debugging

Suppose you need the two fundamental solutions of the second order differential equation

$f'' + xf' + f = 0$ where ' denotes differentiation by x ; use Maple's differential equation solving command **dsolve**

> **restart;**

> **dsolve(diff(f(x),x\$2)+x*diff(f(x),x)+f(x)=0);**

$$f(x) = \frac{\operatorname{erf}\left(\frac{1}{2}\sqrt{2}x\right) - C1}{e^{\left(\frac{1}{2}x^2\right)}} + \frac{-C2}{e^{\left(\frac{1}{2}x^2\right)}}$$

>

The variable $-C1$ is the arbitrary multiplier of the first fundamental solution and $-C2$ is the multiplier of the second. It involves a function $\operatorname{erf}(x)$ that you haven't encountered yet, but you will. Solving differential equations is a big subject and we will devote all of Chapter 7 to it.

Problem 1.17

Find the solution of the differential equation $y' = x+y$. After Maple finds the general solution, find the value of the arbitrary constant from the condition $y(0)=1$. Put a comment about what you are doing in a paragraph of text below and outside the execution group.

Maple can also handle the initial conditions for you, like this (using $f(0) = 1$ and

$$\left(\frac{\partial f}{\partial x}\right)(0) = 2 \text{)}:$$

> **Eq:=diff(f(x),x\$2)+x*diff(f(x),x)+f(x)=0;**

$$Eq := \left(\frac{\partial^2}{\partial x^2} f(x)\right) + x \left(\frac{\partial}{\partial x} f(x)\right) + f(x) = 0$$

> **dsolve({Eq,f(0)=1,D(f)(0)=2},f(x));**

$$f(x) = -\frac{I \operatorname{erf}\left(\frac{1}{2} I \sqrt{2} x\right) \sqrt{\pi} \sqrt{2}}{e^{\left(\frac{1}{2} x^2\right)}} + \frac{1}{e^{\left(\frac{1}{2} x^2\right)}}$$

>

Handle matrices and solve systems of equations

Debugging

There is an entire package in Maple devoted to matrices, vectors, and linear algebra. We will spend Chapter 5 on this subject, but here is an example of what can be done. Let's solve the linear system of equations

$$x + 2y + 3z = 6$$

$$4x + 5y + 6z = 15$$

$$7x + 8y + 10z = 25$$

which has been carefully built so that its solution is $[x,y,z] = [1,1,1]$. Maple would solve this system this way. First bring in the linear algebra package by using the command `with(LinearAlgebra)`. Then load the matrix with the `Matrix` command, load the right-hand side vector with the `Vector` command, and finally solve the system with the `LinearSolve` command.

> **restart;**

> **with(LinearAlgebra);**

[Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, ConditionNumber, ConstantMatrix, ConstantVector, CreatePermutation, CrossProduct,

DeleteColumn, DeleteRow, Determinant, DiagonalMatrix, Dimension, Dimensions, DotProduct, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, GaussianElimination, GenerateEquations, GenerateMatrix, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, LA_Main, LUDecomposition, LeastSquares, LinearSolve, Map, Map2, MatrixAdd, MatrixInverse, MatrixMatrixMultiply, MatrixNorm, MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial, Minor, Multiply, NoUserValue, Norm, Normalize, NullSpace, OuterProductMatrix, Permanent, Pivot, QRDecomposition, RandomMatrix, RandomVector, Rank, ReducedRowEchelonForm, Row, RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector, SchurForm, SingularValues, SmithForm, SubMatrix, SubVector, SumBasis, SylvesterMatrix, ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip]

(If you don't want to see all of this stuff that printed because we pulled in Maple's linear algebra package, put a colon at the end instead of a semicolon.)

> **A:=Matrix([[1,2,3],[4,5,6],[7,8,10]]);**

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

> **b:=Vector([6,15,25]);**

$$b := \begin{bmatrix} 6 \\ 15 \\ 25 \end{bmatrix}$$

> **c:=LinearSolve(A,b);**

$$c := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

>

Problem 1.18

Jack is twice as old as Jill. Jill's age added to Jack's age is the same as Joan's age. The sum of their three ages is 48. How old is each person? Solve this problem by copying and modifying the above example. Put a comment about what you are doing between the **Vector** and **LinearSolve** execution groups. If your copy of the code above doesn't have these in separate execution groups, split them by looking in the edit menu.

Do algebra

[Debugging](#)

This is a very big topic and one of the main things Maple was designed to do. Chapter 9 has a description of common algebra commands that you will be using throughout this course. You can use Chapter 9 as a reference whenever Maple gives you an answer whose form doesn't impress you. Take a few minutes now and read through Chapter 9, down to the section on algebra with matrices. Don't try to understand in detail everything that you read; just become familiar with what's available in Chapter 9 so you know where to look when problems come up later. Then come back here and do the problem below. (Don't do the problems at the end of Chapter 9.) [Chapter 9](#)

Problem 1.19

Use Maple commands to put the expression $\frac{1}{x} - \frac{x}{x-1} + \frac{x}{x+1}$ into a simpler form. Put a comment about what you are doing in text above and outside the first execution group.

[Go to top of section](#)

[Go to top of section](#)

[Go to top of chapter](#)

>

>

[Chapter2.mws](#)

- [Making x-y plots](#)
- [Basic function review](#)
- [Fourier analysis](#)
- [Plotting data](#)
- [Parametric plots](#)
- [Special functions](#)
 - [Problem 2.13: Legendre polynomials](#)
 - [Problem 2.14: Bessel functions](#)
 - [Problem 2.15: The Gamma function](#)
 - [Problem 2.16: Complete elliptic integrals](#)
 - [Problem 2.17: The error function](#)
- [Animations](#)
- [Visualizing vector fields](#)
- [3D-surface plots and contour plots](#)

Chapter 2: Plotting

Maple is a great tool for making plots. In this chapter we will learn how to make plots and then use them to review some important mathematical ideas that are important in physics.

Making x-y plots

[Debugging](#)

The simplest kind of plot is the 2-d, or x-y, plot and Maple does them with ease. Maple will plot both functions and expressions, but the syntax is different for each of these two data types. Here's a plot of the

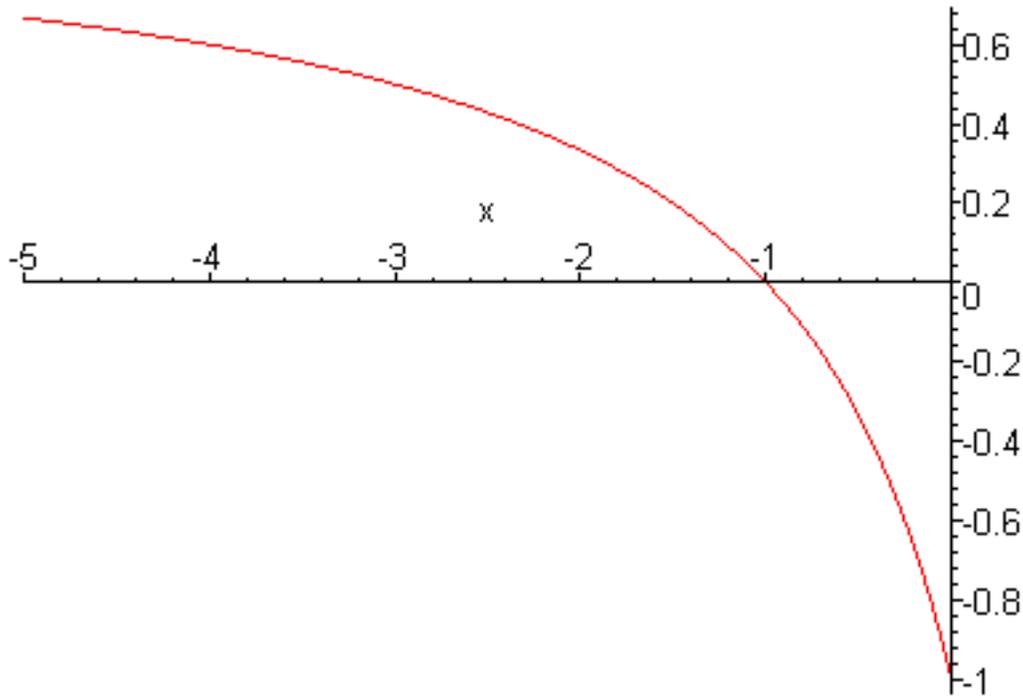
rational expression $\frac{x+1}{x-1}$ from $x = -5$ to $x = 0$. First we assign the rational expression to the

variable g , then plot g . Note that g is an expression, not a function, so you can't use the notation $g(x)$ to plot it.

```
> g:=(x+1)/(x-1);
```

$$g := \frac{x+1}{x-1}$$

> **plot(g,x=-5..0);**



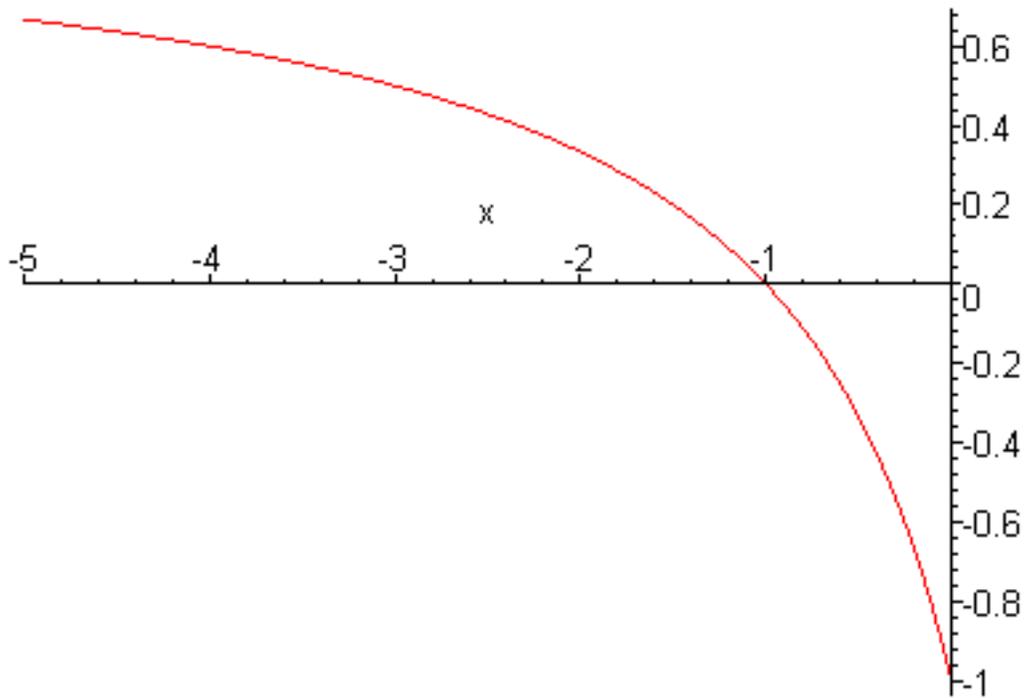
This same plot can be obtained using Maple's function notation, like this (I have purposely made the function have two arguments so you can see how to plot such a function)

> **f:=(x,y)->(x+y)/(x-y);**

$$f := (x, y) \rightarrow \frac{x+y}{x-y}$$

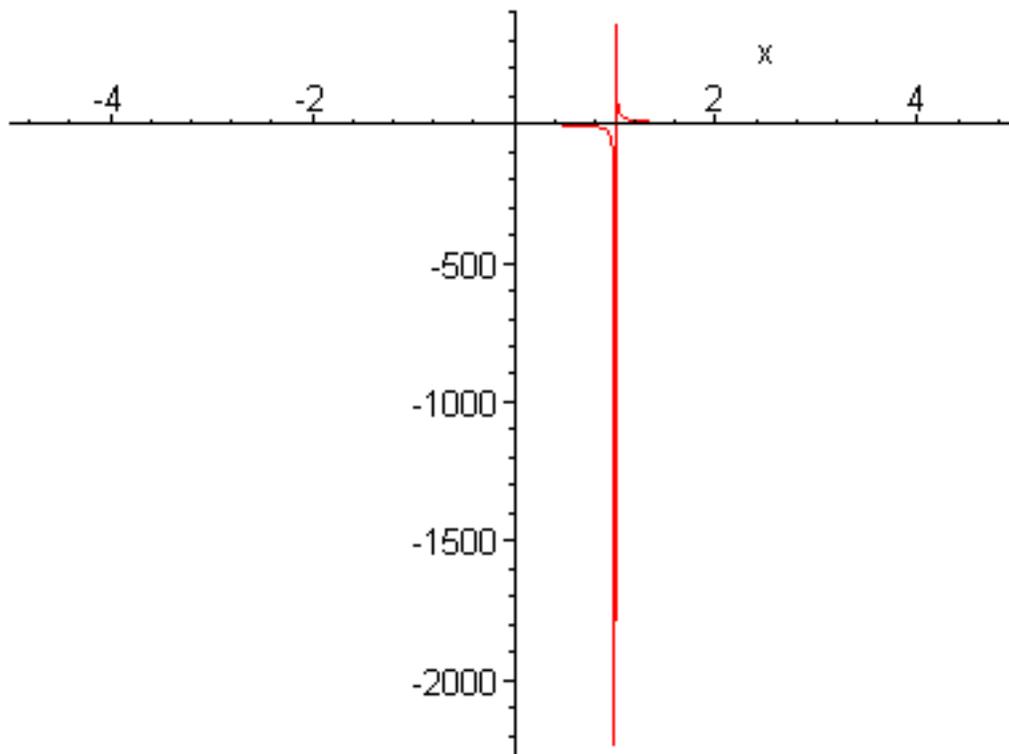
To make a plot in x just give y a value through the argument list ($y = 1$ reproduces the previous plot) and plot in x

> **plot(f(x,1),x=-5..0);**



OK, now let's plot this same function over a larger range. Notice that you can just put the expression in the plot command line and skip the assignment of it to a variable if you want.

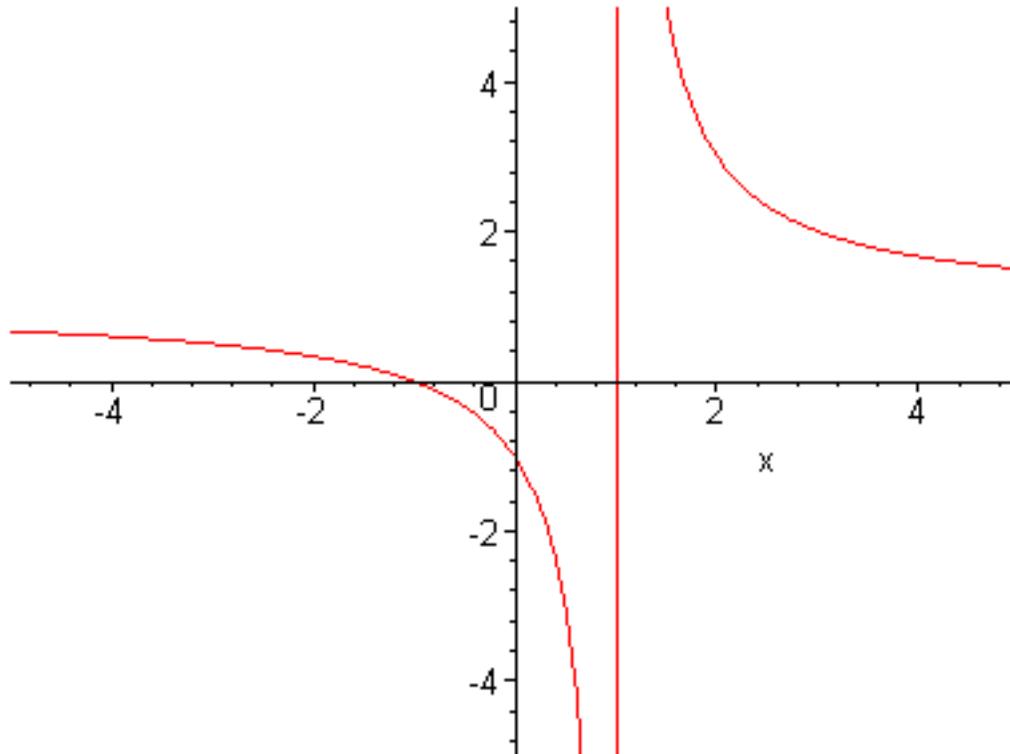
```
> plot((x+1)/(x-1),x=-5..5);
```



Well, it's a plot, but perhaps not the one we wanted. The problem is that our function is singular at

$x = 1$, so Maple tried to give us its huge vertical range. We would probably rather not try to visualize infinity, so we would like to limit the vertical range. This is easy too.

```
> plot((x+1)/(x-1),x=-5..5,-5..5);
```

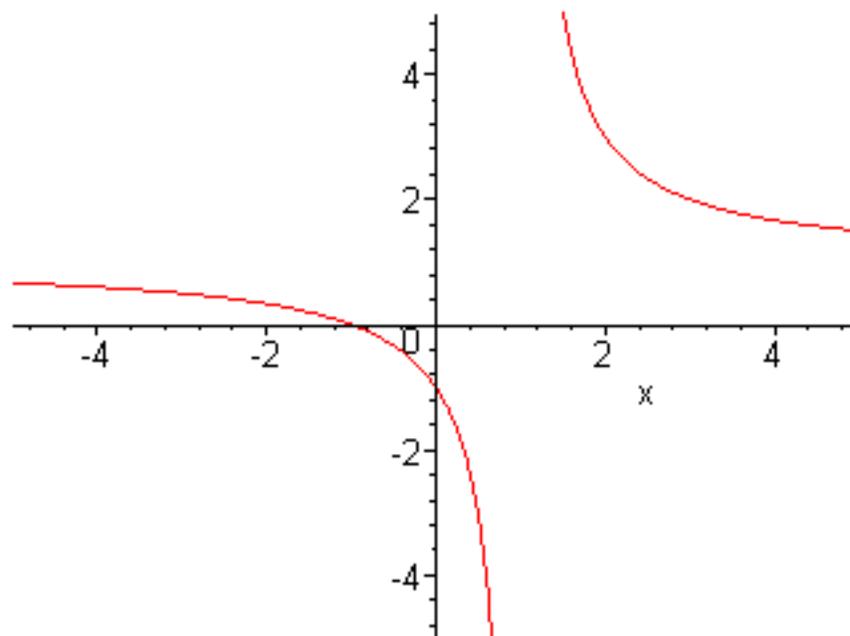


Now this looks better, but the funny-looking vertical line at $x = 1$ looks wrong, and it is. Maple just connected the dots, including the jump from $-\infty$ to ∞ . To clean this up we have to work a little harder and give the plot command some options. There are many options you can specify and to become an expert plotter you have to be able to look them up and use them. You can find them by using

```
> ?plot[option]
```

and the one we need to make the plot above look better is the **discont=true** option, which tells Maple to look for discontinuities in our function and turn off dot-to-dot plotting.

```
> plot((x+1)/(x-1),x=-5..5,-5..5,discont=true);
```

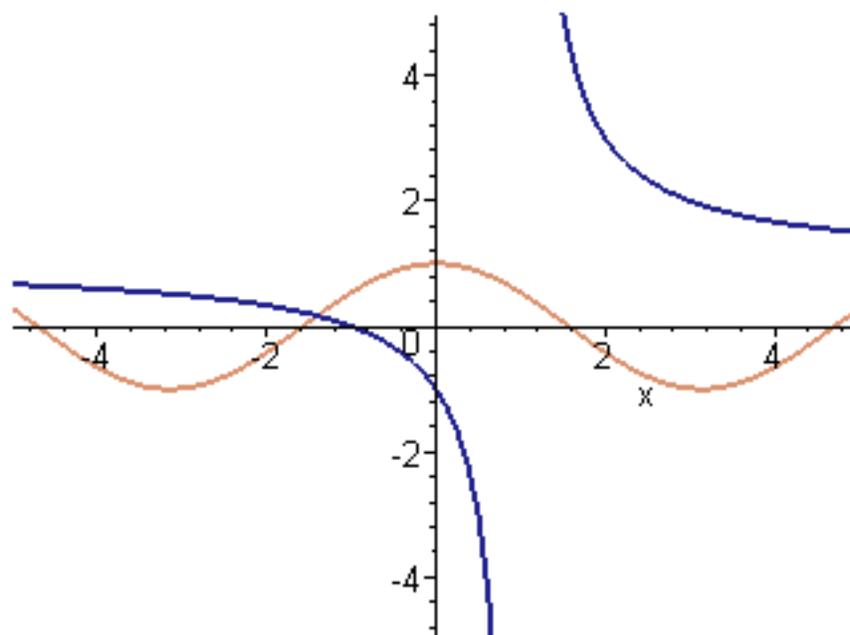


Now let's try something a little fancier; let's put two plots on the same frame and specify the two different colors we want to use to tell them apart. You can see what colors are available by using

> **?plot[colors]**

Let's use the school colors of Brigham Young University, as far as they are translated correctly by Maple, and also will thicken the lines to make the colors stand out. (You might want to use thicker lines like this on a plot so that it would be of publication quality.)

> **plot([(x+1)/(x-1),cos(x)],x=-5..5,discont=true,color=[navy,tan],thickness=2);**



As you can see, the two functions are placed in a Maple list, which is a list of things separated by commas and contained within square brackets. Any options that apply separately to each plot are also specified in lists, as you can see in the **color** option. *Warning: Maple does not always respect your color order. If you have lots of plots and lots of colors, Maple sometimes mixes the colors up.*

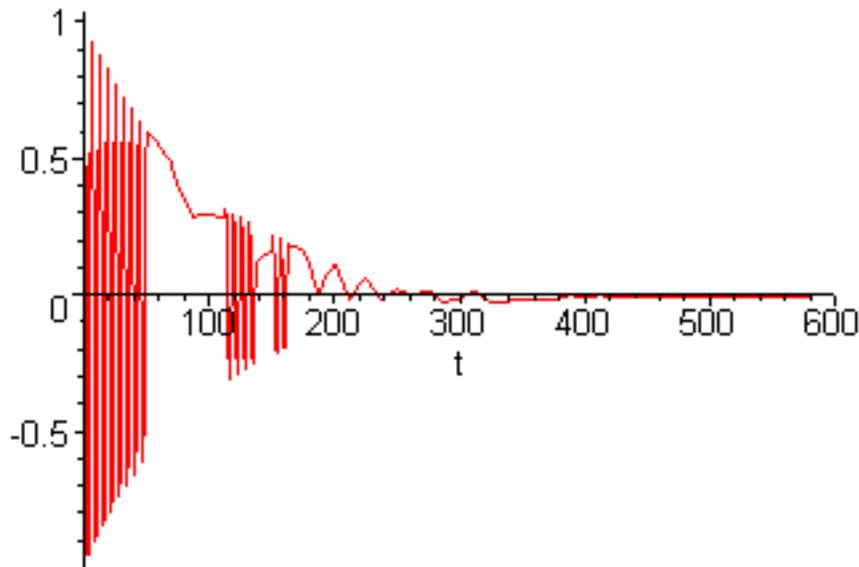
>

I'll give you one more sample plot and then you will be off to the exercises. Suppose you wanted to

make a plot of the motion of a damped harmonic oscillator, given by the formula $e^{-\frac{t}{100}} \cos(t)$

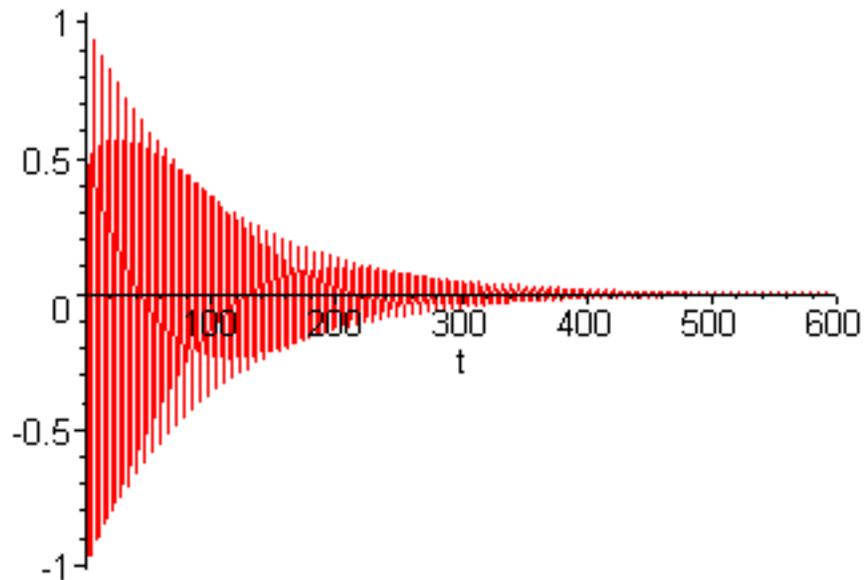
from $t = 0$ to $t = 600$. Since you now know how to plot you would just use

> **plot(exp(-t/100)*cos(t),t=0..600);**



and you would be pretty sure that something had gone wrong. What went wrong is that Maple is not smart enough on its own to know how many points to plot; so when a function is wiggling like crazy, like this one is, it needs some help from you. You can fix it by using the **numpoints** option, like this

> **plot(exp(-t/100)*cos(t),t=0..600,numpoints=400);**



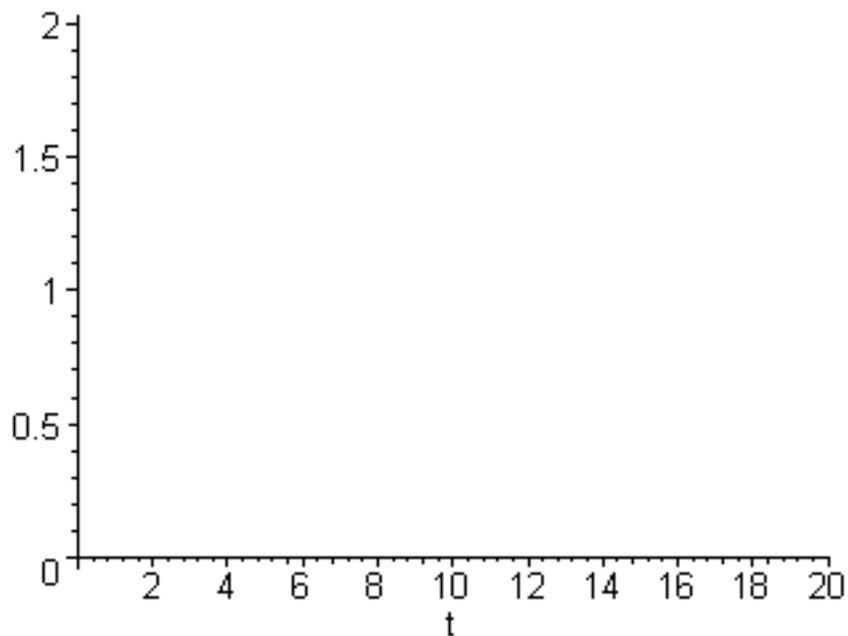
One last, but very important, thing; you will often want to define your own expressions to plot using Maple's assign command (`:=`), like this:

```
> F:=cos(omega*t)^3;
```

$$F := \cos(\omega t)^3$$

Since it clearly depends on t we should be able to plot it

```
> plot(F,t=0..20);
```



This blank plot is actually a Maple error message. The problem is that ω doesn't have a value yet, so there is no way Maple could plot it. But how are you supposed to know this? Try giving t a value and evaluating F .

```
> t:=7.;F;
```

```
t := 7.
```

```
cos(7.  $\omega$ )3
```

The ω still sitting there is your clue; it needs a value. So let's assign it a value and try again.

```
> omega:=1.;plot(F,t=0..20);
```

```
 $\omega$  := 1.
```

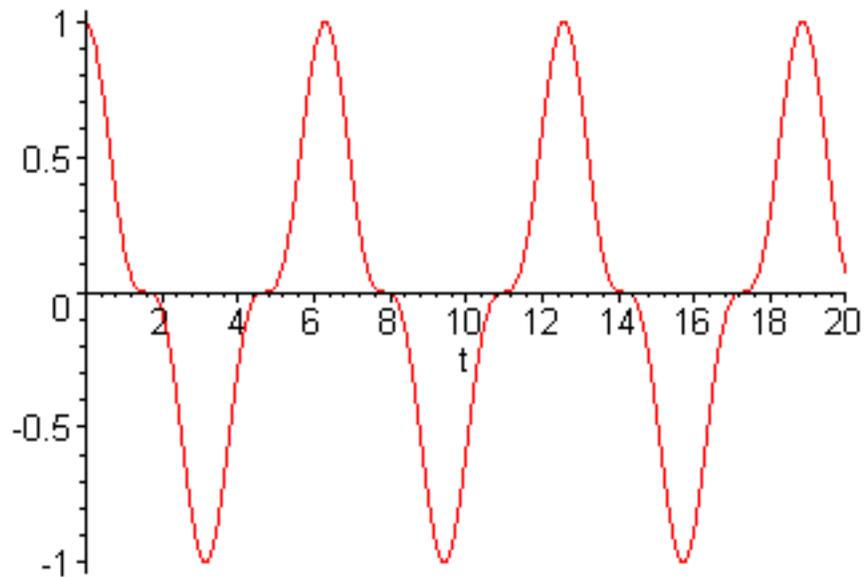
```
Error, (in plot) invalid arguments
```

What now!! The problem is subtle, and it will bite you often when you debug plots this way. Remember that we gave t a value of 7 so we could see what's wrong? Maple remembers this assignment and

won't let us use t as a running variable in the plot. To fix it we have to undo the setting of t to 7 with the `unassign` command with 't' as its argument, then try it again.

```
> unassign('t');omega:=1;plot(F,t=0..20);
```

```
omega := 1
```

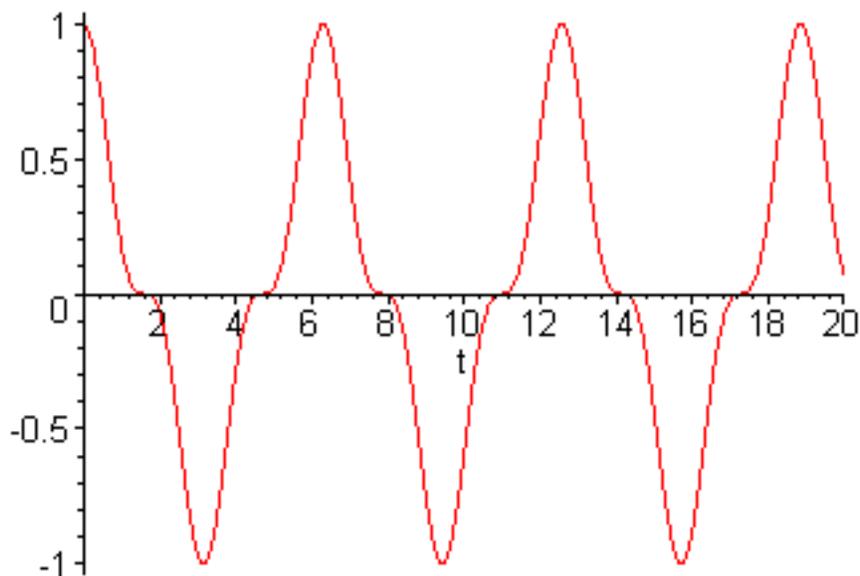


or you could use the shorthand version of the `unassign` command like this

```
> t:='t';omega:=1;plot(F,t=0..20);
```

```
t := t
```

```
omega := 1
```

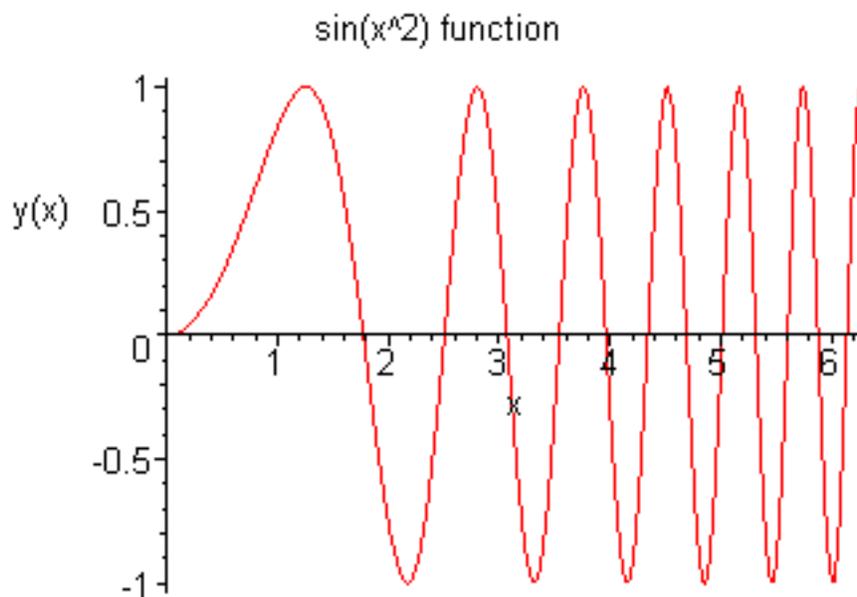


The way to read this is "assign t to its own name, erasing any previous assignments."

>

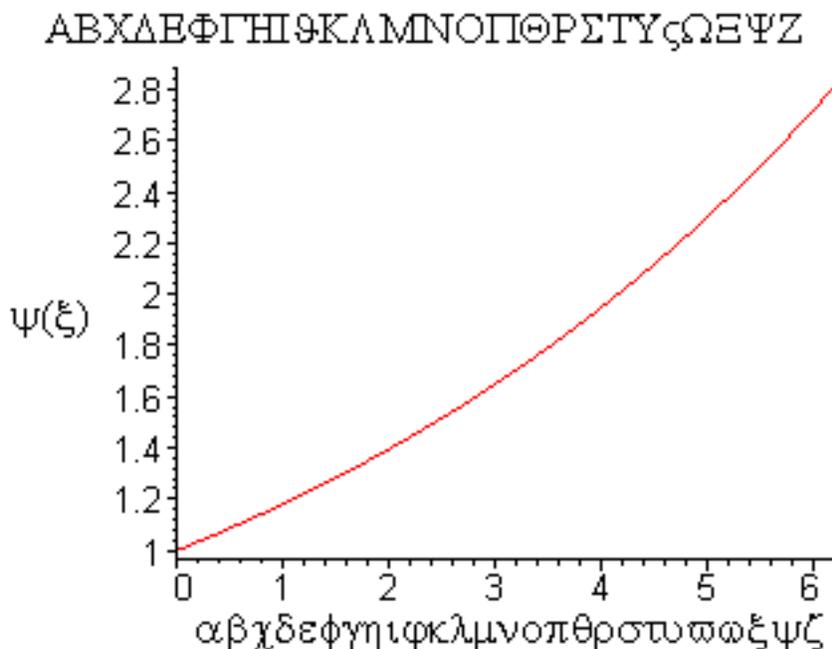
Before we leave the introduction let's look at one more thing you will want to know so that you can make your plots look nice: lettering. To put labels on the axes of a plot you can use the labels option, like this: `labels=["x","y"]`. To put a title on use `title="This is the title"`. And if you want to put text in the figure somewhere, look up `textplot` in online help. Here's a plot illustrating these options.

> `plot(sin(x^2),x=0..2*Pi,labels=["x","y(x)"],title=" sin(x^2) function");`



Notice that $\sin(x^2)$ didn't print in Maple's pretty math mode. I wish I knew how to make this happen in plots. But Greek letters are available, in a sort of limited form, by changing either the labelfont or the titlefont to SYMBOL, like this (I have purposely dumped the entire greek alphabet on the plot so you can see how they are coded)

```
> plot(exp(x/6),x=0..2*Pi,labels=["abcdefghijklmnopqrstuvwxy", "y(x)"],
title="ABCDEFGHIJKLMNPOQRSTUVWXYZ",labelfont=[SYMBOL],titlefont=[SYMBOL]);
```



Unfortunately, I can't tell you how to put labels and titles on that are mixtures of Roman and Greek letters. What Maple really needs is a text production capability like LaTeX where full typesetting of text and equations is possible. Perhaps in a later version.

>

Ok, off to the exercises in the subsections below. Along the way you will learn some more plotting tricks.

[Go to top of section](#)

Basic function review

[Debugging](#)

In this section we will review the basic functions of mathematical physics by making plots.

Problem 2.1

Using a horizontal range of -2π to 2π and a vertical range of -2 to 2, plot the functions $\sin(x)$, $\cos(x)$, and $\tan(x)$. Plot all three on the same plot by putting these three expressions in a list in the plot command line. Choose special colors for each one.

Problem 2.2

Using a horizontal range of -3 to 3 and a vertical range of -3 to 3, plot the functions $\sinh(x)$, $\frac{e^x}{2}$, and $\frac{e^{-x}}{2}$. Choose special colors for each one and study the plot carefully so that you can

remember how they relate. Now on another picture plot the functions $\frac{e^x}{2}$ and $\frac{e^{-x}}{2}$ together with $\cosh(x)$. Finally, plot the functions $\sinh(x)$, $\cosh(x)$, and $\tanh(x)$ on the same picture and stare at them until you are sure you will remember what they look like.

Problem 2.3

Verify by drawing the appropriate graphs that the following identities are true. (This is not the same as a proof, of course, but if you are a person who thinks visually, a graph is worth a thousand lemmas.)

$e^x e^{(2x)} = e^{(3x)}$; $\ln(x) + \ln(2x) = \ln(2x^2)$; $5 \ln(x) = \ln(x^5)$. To avoid taking the

logarithm of negative numbers make the plot only over positive values of x . Note: it will be easier to see that the two expressions are graphically equivalent if you plot them with different colors and

thicknesses by using this option line in your plot command: **color=[red,green],thickness=[2,10]**

Problem 2.4

Show by doing graphical experiments that the exponential function grows faster than any power of x and that the natural log function grows more slowly than any fractional power of x . Do this by choosing several fixed powers of x and then making plots over appropriately sized windows. Recall that if you leave the vertical scale out of the plot command, Maple will choose it appropriately. Note: when you do the exponential function you will probably run into a floating point error because you are trying to plot numbers that are too big. The standard way to avoid this problem is to plot the log of the functions instead. It is easier to tell how big the numbers are if you use the function $\log_{10}(x)$. Second note: when you do the fractional power comparison you will have to use such big values of x that Maple will complain. So in place of x use $\log_{10}(x)$, i.e., use $x=10^t$ and make the plots versus t instead of x .

Problem 2.5

Graphically verify the following trigonometric identities. Instead of overlaying two plots, plot the left-hand side minus the right-hand side. Comment on why you don't really get zero.

$$\sin(2x) = 2 \sin(x) \cos(x) \quad ; \quad \cos(2x) = \cos(x)^2 - \sin(x)^2 \quad .$$

$$\sin\left(\frac{x}{2}\right)^2 = \frac{1 - \cos(x)}{2} \quad ; \quad \cos\left(\frac{x}{2}\right)^2 = \frac{1 + \cos(x)}{2}$$

Problem 2.6

Show graphically that when you add $\sin(x)$ and $\cos(x)$ together you just get another sinusoidal function with a new amplitude and phase shift. Make the plot by assigning the sum of these two functions to a Maple function F and then plotting $F(x)$. From your graph find the new amplitude and the phase shift relative to the cosine function, i.e., find a formula for the addition of sine and cosine in the

form $A \cos(x - \phi)$.

[Go to top of section](#)

Fourier analysis

[Debugging](#)

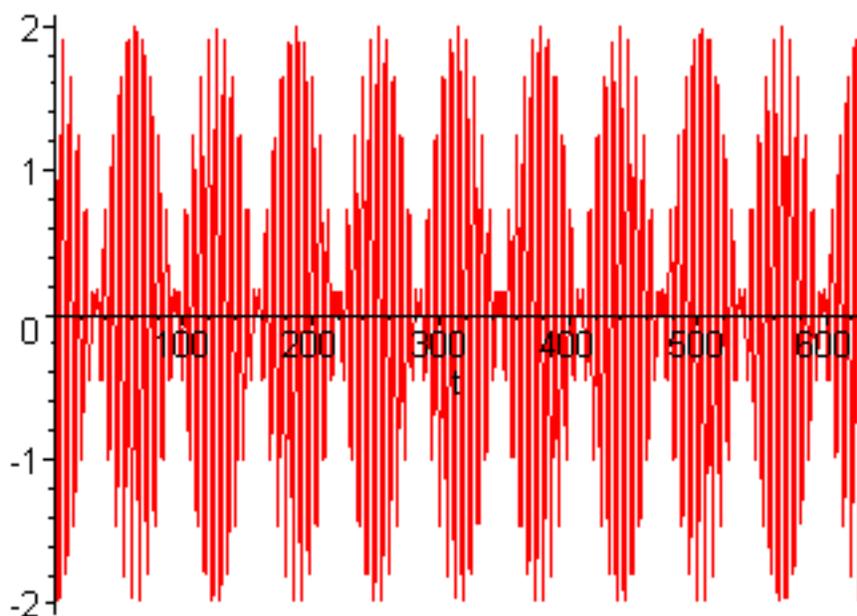
Later on you will learn some powerful techniques for solving partial differential equations using a technique called *Fourier analysis* , which is based on this powerful idea: you can make complicated functions by using infinite sums of simple ones. In the previous section you saw an example of this when you added cosine and sine to get a phase-shifted cosine. Here's another simple example that is important in physics. Add two cosine functions, but with different frequencies:

> **restart;**

> **f:=t-> cos(t)+cos(1.1*t);**

$$f := t \rightarrow \cos(t) + \cos(1.1 t)$$

> **plot(f(t),t=0..200*Pi,numpoints=1000);**



You have probably experienced this plot. Remember what the trumpets in junior high band sounded like: wah-wah-wah-wah. The annoying flutter of two inexperienced musicians trying to play the same note, but not quite succeeding, is called beats, and they are clearly illustrated by the plot above. The rapid oscillation is the note and the successive amplitude changes are the flutter.

>

Problem 2.7

Play with the graph of beats above and find the relation of the difference between the two frequencies ($\omega_1 = 1$ and $\omega_2 = 1.1$ in the example above) to the wah-wah frequency, defined by

$f = \frac{1}{T_{beat}}$, where T_{beat} is the time between successive loud portions of the sound.

Now let's illustrate Fourier's theorem. He showed that any periodic function could be represented by an infinite sum of sines and cosines having the same period as the function. In these sums the sines and cosine functions are multiplied by coefficients that usually get small as the sum index gets big. For

instance, here's a sum of cosine functions with coefficients that fall off like $\frac{1}{n}$ coded using Maple's

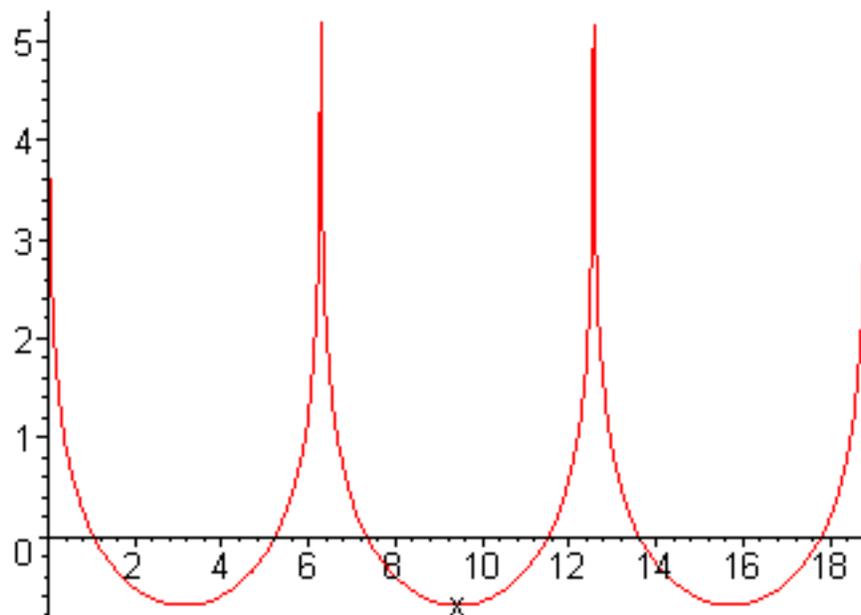
sum command.

> **g:=x->sum(cos(n*x)/n,n=1..100);**

$$g := x \rightarrow \sum_{n=1}^{100} \frac{\cos(n x)}{n}$$

There isn't any way you can get any insight into what this function looks like by staring at this sum, but Maple's plotting capability can show you exactly what is going on

> **plot(g(x),x=0..6*Pi);**



Notice (i) that the function is periodic and (ii) that you probably couldn't have guessed that it was going to look like this. You can do some more examples in the problem below.

>

Problem 2.8

Consider the Fourier sum $\sum_{n=1}^{100} \frac{(-1)^n \cos((2n-1)t)}{2n-1}$. Plot it from $t=0$ to $t=2\pi$ by

assigning this sum to a Maple function. Now modify the sum so the factor in the denominator is squared, switch cosine to sine, and plot it again. Then change the factor in the denominator to be cubed and plot again. Then switch to the 4th power and plot again. This sequence illustrates a property of Fourier series which isn't in most math books. Functions which are discontinuous have Fourier series in which the

terms go to zero like $\frac{1}{n}$. (And at the discontinuities they have nasty wiggly features called the Gibbs

phenomenon, which has someone's name attached to it because it had to be discovered the hard way, Maple plots having not yet been invented.) Functions which have discontinuous first derivatives fall off

like $\frac{1}{n^2}$. Functions which have discontinuous second derivatives fall off like $\frac{1}{n^3}$, etc.. And

functions which have no discontinuities or discontinuous derivatives of any order have to fall off faster

than any power of $\frac{1}{n}$, which means they have to fall off exponentially in n , like $e^{(-n)}$,

$\frac{1}{\cosh(n)}$, etc.. You will see Fourier series that fall off like this in Physics 441.

Advanced topic: wave packets

Finally, we can use the idea of Fourier analysis to build wave packets and to examine the uncertainty principle. You probably think of the uncertainty principle only in terms of quantum mechanics, but it is really a result of Fourier analysis. But since Fourier analysis underlies all of quantum mechanics, it should be no surprise that this important physical principle turns out to be a mathematical result.

To build a wave packet we sum a whole bunch of cosines or sines with frequencies closely spaced around some central frequency. The result is a wave at the central frequency but it lives inside an envelope caused by the destructive interference of the neighboring waves. Here, I'll show you.

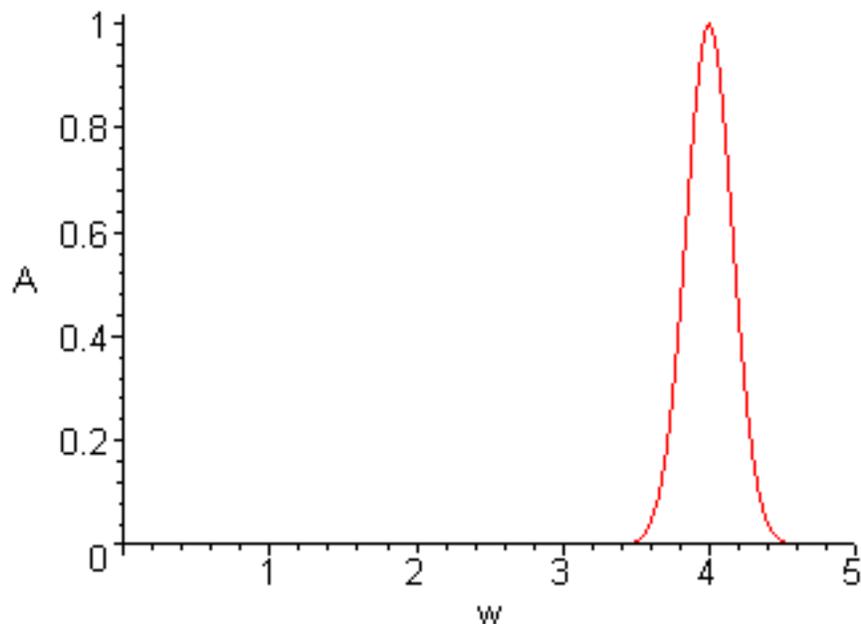
We'll use a Gaussian centered at $\omega = 4$ as the distribution of frequencies in the packet

```
> A:=w-> exp(-(w-4)^2*20);
```

$$A := w \rightarrow e^{(-20 (w - 4)^2)}$$

Let's plot it so you can see the distribution of frequencies

```
> plot(A(w),w=0..5,labels=['w','A']);
```



It is important to have a way of describing how wide this distribution is. A common term that you may have learned before is the Full Width at Half Maximum, or FWHM for short. This is the width in ω between the two points in the distribution $A(\omega)$ where A is one-half of its maximum. Look at the

graph and roughly verify that the FWHM is given by $\Delta \omega = 2 \sqrt{\frac{\ln(2)}{20}}$. Then do a little math and

see that this formula is correct.

Now we will build a function of time which is made up of a superposition of waves with frequencies between $\omega = 3.5$ and $\omega = 4.5$ (frequencies outside of this range don't have enough amplitude to matter very much, according to the graph above).

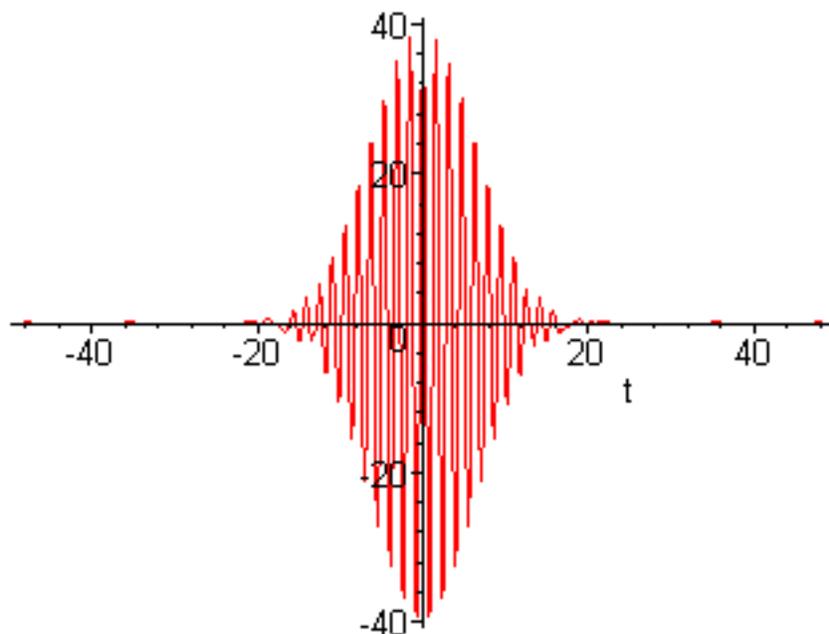
> **f:=t-> sum(cos((3.5+n/100)*t)*A(3.5+n/100),n=0..100);**

$$f := t \rightarrow \sum_{n=0}^{100} \cos\left(\left(3.5 + \frac{1}{100}n\right)t\right) A\left(3.5 + \frac{1}{100}n\right)$$

Note that as n ranges from 0 to 100, the quantity $3.5 + \frac{n}{100}$ ranges from 3.5 to 4.5, just as desired

for ω . Now we can plot the time signal and see what time function $f(t)$ is produced by the frequency function $A(\omega)$.

> `plot(f(t),t=-50..50);`



So we see that a finite-width distribution of amplitudes in frequency space (ω) makes a finite-width distribution in time, which is exactly what a wave packet is. Now comes the uncertainty principle. This theorem says that the product of the width of the amplitude function $A(\omega)$, which we call $\Delta \omega$, and the width of the time signal Δt (the width of the wave packet plotted above) is no smaller than a number of order unity. In practice it is usually about equal to 5.5 if we measure the widths using the full-width at half-maximum. Or in symbols

$$5.5 \leq \Delta \omega \Delta t$$

>

Problem 2.9

Verify the uncertainty principle given above for the graphs displayed. Then change the width of the amplitude function $A(\omega)$ by changing the parameter 20 in the definition of $A(\omega)$ to something else and making a new set of plots. Verify that the uncertainty relation holds for the new plots.

[Go to top of section](#)

Plotting data

[Debugging](#)

Maple is not really a very good way to do data analysis. You have learned, or will learn, some powerful ways to do data analysis when you study Labview in your physics lab courses or when you study Matlab in Physics 330. But you might want to import some data into Maple sometime and compare it with a formula that you have coded in Maple, so here's what you need to know.

Maple stores and plots data in matrix form. The way to read data in is to use `readdata`, like this

```
> d:=readdata("datafile",2);
```

```
Error, (in readline) file or directory does not exist
```

where the 2 means that there are 2 columns of data in the file. If you execute this command you will get an error because you don't have a file called datafile in the directory you are working in. To fix this problem use Window's notepad editor to create a file called datafile that looks like this: (note that you can copy and paste this data into notepad)

```
>
```

```
0 1.0
```

```
1 1.5
```

```
2 1.9
```

```
3 2.6
```

```
4 3.7
```

```
5 5.4
```

```
6 7.3
```

```
7 10.4
```

8 14.2

9 20.0

10 29.0

After you have made this file and have properly saved it in the directory you are working in, try executing the command again.....did it fail? If so, it may be because Notepad insists on putting a .txt extension on all of its files. So the **readdata** command may need to be changed to

```
> d:=readdata("datafile.txt",2);
```

```
d := [[0., 1.], [1., 1.5], [2., 1.9], [3., 2.6], [4., 3.7], [5., 5.4],
[6., 7.3], [7., 10.4], [8., 14.2], [9., 20.], [10., 29.]]
```

This still might not work if you are on a windows system. The problem now is that Maple is not looking for your file in the right directory. To fix this use the **currentdir** command: `currentdir("directory name")`, e.g.,

```
> currentdir("c:\\Physics 230");
```

```
Error, (in currentdir) file or directory does not exist
```

Note that you can use this command in this form to find out what the current directory is

```
> currentdir();
```

```
"C:\\Program Files\\Maple 7"
```

Or if you are running off the a: drive you would use

```
> currentdir("a:");
```

```
Error, (in currentdir) file or directory does not exist
```

(Note that this change-directory command has the annoying feature in Windows that instead of echoing the new directory it echos the old one, or perhaps it doesn't echo anything. To make sure you are where

you want to be check it with `currentdir()` .) Once you get this straight one of the two read commands above should work.

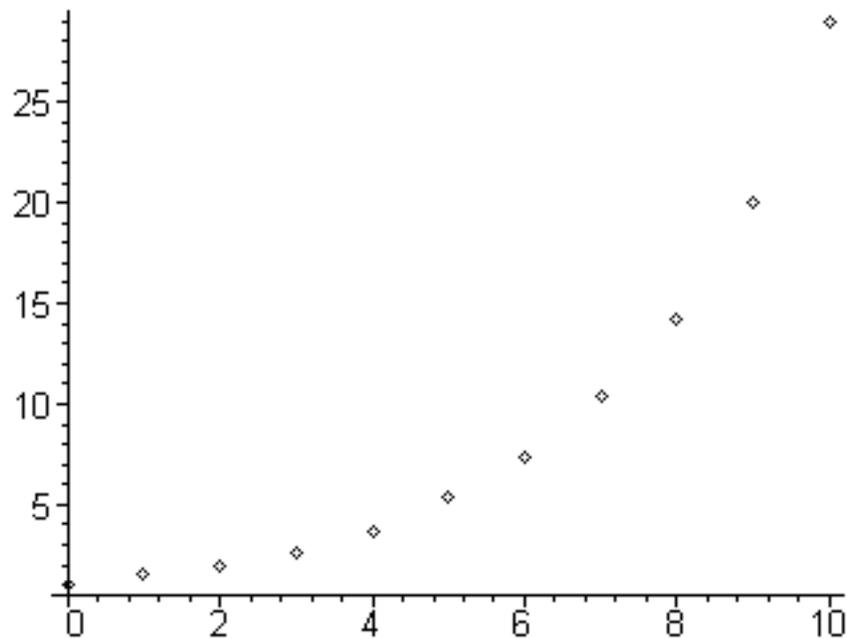
>

Now that the data is stored as a matrix Maple can plot it using the `pointplot` command, as shown below. But if you try to execute the `pointplot` command it will just print a pretty blue version of the command and ignore you. The problem is that the package that contains `pointplot` is so big that Maple doesn't load it unless it is needed, so you have to tell it that it's needed by using `with(plots)`

> **with(plots):**

Warning, the name `changecoords` has been redefined

> **pointplot(d);**



>

Notice that the `with(plots)` command was terminated with a `:` instead of a `;`. The `:` means to execute the command just like `;` but to shut up about it. Try doing `with(plots);` and you will see what I mean. Whenever you want to execute a command and you are so confident that it will work that you don't need to see the output, use `:. (You should rarely be this confident. Only use : after you have tested the`

$$\left(\frac{x}{3}\right)$$

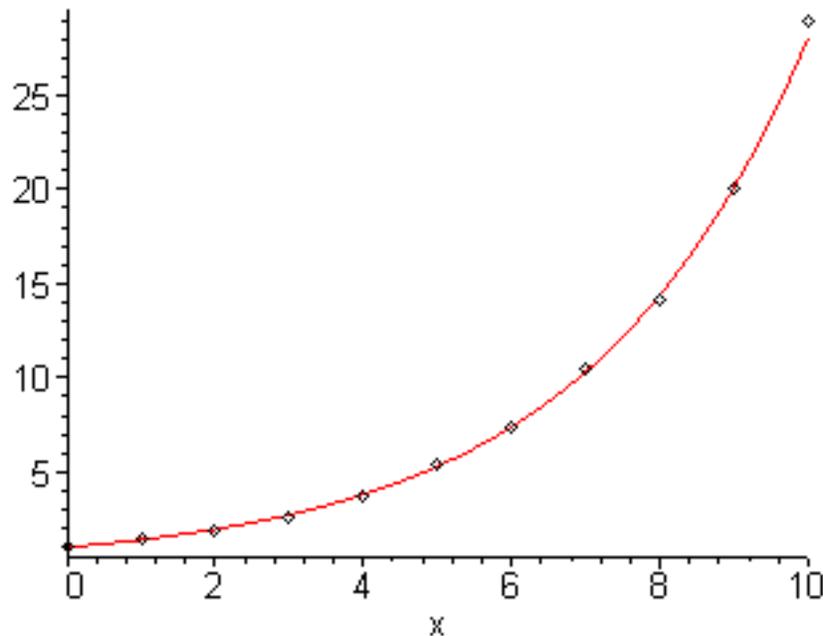
command with ; first.) Now suppose we want to see how well this data is fit by the formula $e^{\frac{x}{3}}$.

We simply assign the point plot to one variable name and the formula plot to another, then display them together, like this (notice the colons at the end of the assign statements--it's better to have Maple shut up about what it is doing here.)

> **p1:=pointplot(d):**

> **p2:=plot(exp(x/3),x=0..10):**

> **display({p1,p2});**



Maple's stats package does similar things, and will even do some fitting.

>

[Go to top of section](#)

Parametric plots

[Debugging](#)

One of the best ways to make complicated plots is with a parametric representation. Consider, for example, the humble circle. The formula for it that you remember best is probably this one

$$x^2 + y^2 = R^2 \quad ,$$

but if you try to graph a circle with this formula you have to do it this way

$$y(x) = \sqrt{R^2 - x^2} \quad \text{and} \quad y(x) = -\sqrt{R^2 - x^2} \quad .$$

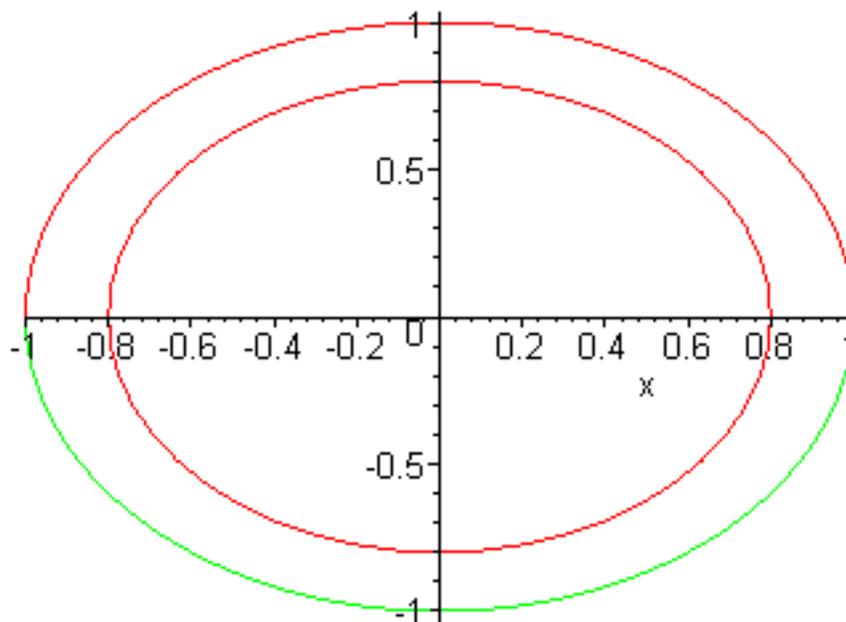
This requires two separate plots and the

functions have infinite derivatives at $x = R$ and $x = -R$; it's just messy. But look how pretty the formulas for x and y on a circle look if you use a parametric representation:

$$x(s) = R \cos(s) \quad ; \quad y(s) = R \sin(s) \quad ; \quad s = 0 .. 2\pi \quad .$$

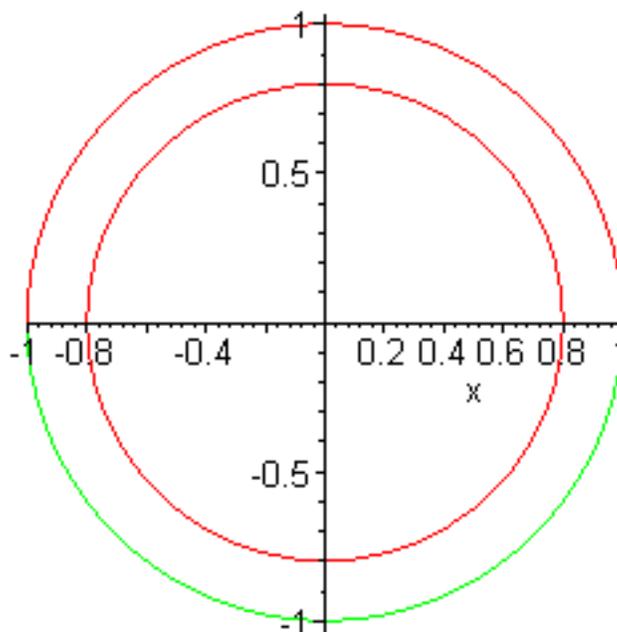
Here are the Maple plot commands to do the plot of a circle both the $y(x)$ way and the parametric way, with the parametric one at a smaller radius so you can see them both.

```
> with(plots):
> p1:=plot([sqrt(1-x^2),-sqrt(1-x^2)],x=-1..1):
> p2:=plot([.8*cos(s),.8*sin(s),s=0..2*Pi]):
> display({p1,p2});
```



Ok--they both look the same, but neither one looks like a circle. This is a common plotting problem in all computer programs: circles look like ellipses because the x and y axes have different scales. Maple has a particularly easy fix for this problem, which is that you just tell the plot command that you want the x and y axes constrained to be the same scale, like this

> **display({p1,p2},scaling=constrained);**



You can explore some more ways of doing parametric plots in the problems below.

>

Problem 2.10

The classic conic sections all have parametric forms. Use the parametric plot command to sketch each of these curves using constrained scaling.

$$\text{Ellipse: } \mathbf{x}(s) = 3 \cos(s) \quad ; \quad \mathbf{y}(s) = 1.3 \sin(s) \quad ; \quad s = 0 .. 2 \pi \quad .$$

$$\text{Hyperbola: } \mathbf{x}(s) = 3 \cosh(s) \quad ; \quad \mathbf{y}(s) = 1.3 \sinh(s) \quad ; \quad s = -1.5 .. 1.5$$

(Once you draw this, figure out how to get the missing half of the picture.)

$$\text{Parabola (rotated): } \mathbf{x}(s) = s \cos(\phi) - s^2 \sin(\phi) \quad ; \quad \mathbf{y}(s) = s^2 \cos(\phi) + s \sin(\phi)$$

$$s = -3 .. 3 \quad .$$

Try plotting this for several different angles. If you want to go a little crazy, try drawing a rotated parabola using the $y(x)$ form.

Problem 2.11

It is also possible to draw curves in polar coordinates. A simple example is the ellipse again, but this time with the origin at one focus of the ellipse. To make sense of this we need some relations from ellipse geometry. Suppose that the (x,y) form of the ellipse is as given above,

$\mathbf{x}(s) = a \cos(s) \quad ; \quad \mathbf{y}(s) = b \sin(s)$ with $b < a$. Then the distance a is called the semi-major axis and the distance b is called the semi-minor axis. The eccentricity of the ellipse is defined to be

$$e = \sqrt{1 - \frac{b^2}{a^2}} \quad . \text{ It is zero for a circle and as it gets smaller the ellipse gets more and more$$

squashed. The ellipse has two foci, one focus to the left of center along the long (semi-major) axis and

the other to the right of center. The distance from the center of the ellipse to either focus is $f = e a$.

(Wait a sec--how come I am dragging you through this? What has this got to do with physics? Well, this is the language you will use in Physics 321 to describe the motion of planets around the sun, so stay with me, this is important.)

Draw an ellipse using an (x,y) parametrization which has the origin of coordinates at the right focus of the ellipse with $a = 2$ and $b = 1$. (Use the standard parametrization given above, but subtract f from $x(s)$ to shift it over.

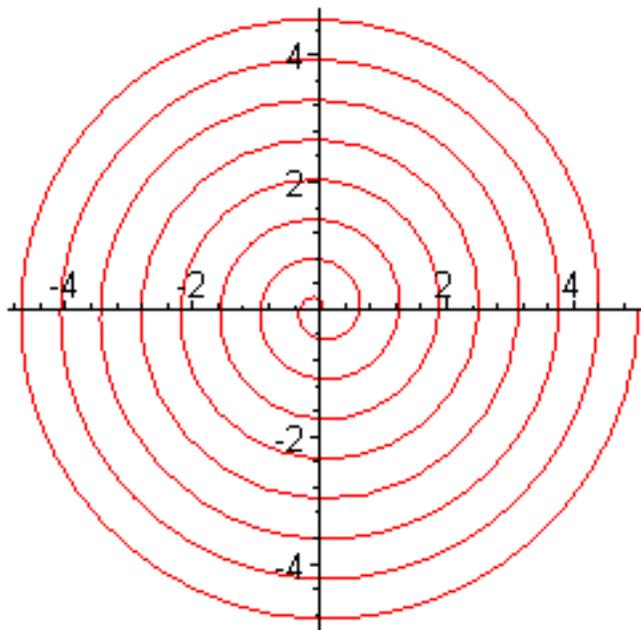
Now, here's the point of this exercise. The parametric form of an ellipse with the origin of coordinates at the right focus in cylindrical coordinates (like you will use in Physics 321) is given by

$$r(\phi) = \frac{a(1 - e^2)}{1 + e \cos(\phi)}$$

. Lay this cylindrical coordinate graph on top of the xy graph of the ellipse

you just did and verify that they match. The form of a Maple parametric plot in cylindrical coordinates is `plot([r(s),phi(s),s=s1..s2],coords=polar,options)`. For example, to make a spiral you could use

> `plot([s/10,s,s=0..16*Pi],coords=polar,scaling=constrained);`



>

Once you get the plots to agree you are finished; but try to remember that the form for an ellipse with long dimension $2a$, eccentricity e , and with the origin at one focus in cylindrical coordinates is

$$r(\phi) = \frac{a(1 - e^2)}{1 + e \cos(\phi)}$$

Ok-you won't remember this. But remember that you can come back here as a review when you get to planetary motion in Physics 321.

Problem 2.12

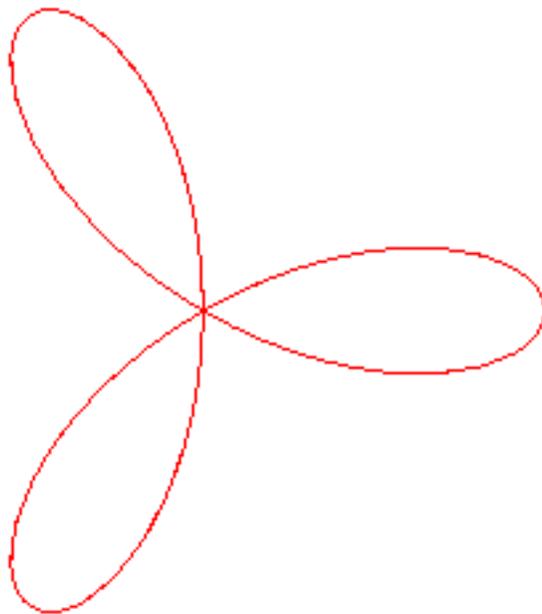
Well, that was pretty bad. Here's one that's just for fun. In the polar plot command given below treat two numbers as adjustable, j and k , and just see what kinds of different pictures you can make. Fractional values of j are interesting if you use large values of k .

> **`j:=3;k:=2;`**

`j := 3`

`k := 2`

> **`plot([cos(j*s),s,s=0..k*Pi],coords=polar,scaling=constrained,axes=none);`**



>

[Go to top of section](#)

Special functions

There are a bunch of functions you haven't encountered yet that are called *special functions*, meaning they are not like the simple sines, cosines, exponentials, logs, powers, and hyperbolic functions you can find on hand calculators. These are functions that arise in more advanced work in mathematical physics and you will work with them in Physics 318, 321, 441, 442, 451, and 452. You can see which functions Maple knows about by using **?inifcns**. Let's get a head start on these functions by using Maple graphs to at least see what they look like. Each problem below introduces you to a different special function and asks you to get a feel for its properties by making graphs.

Problem 2.13: Legendre polynomials

Debugging

These functions show up when we do electrical and quantum mechanical problems in spherical coordinates. Their symbol is $P_n(x)$ and they are simply polynomials in x . But they have very special properties on the interval $[-1,1]$. (i) For odd values of the integer index n they are odd and for even n they are even. (ii) $P_n(1) = 1$ and $P_n(-1) = (-1)^n$. (iii) They wiggle a lot, more and

more as n gets bigger and bigger, but their magnitudes are always smaller than 1 inside the interval $(-1,1)$. I have given you below a plot command that plots the first two on top of each other. Modify this command so that it plots the first 8 Legendre polynomials on top of each other.

You could type all 8 of them in, or you could use Maple's sequence command **seq** which works like this.

Suppose you wanted to generate a sequence of the functions x^n from $n=1..8$. You would do this

> **seq(x^n,n=1..8);**

$$x, x^2, x^3, x^4, x^5, x^6, x^7, x^8$$

You can do this same kind of thing with **P(n,x)** in the plot command below.

> **with(orthopoly);**

$$[G, H, L, P, T, U]$$

> **seq(P(n,x),n=0..8);**

$$1, x, \frac{3}{2}x^2 - \frac{1}{2}, \frac{5}{2}x^3 - \frac{3}{2}x, \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8},$$

$$\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x, \frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16},$$

$$\frac{429}{16}x^7 - \frac{693}{16}x^5 + \frac{315}{16}x^3 - \frac{35}{16}x,$$

$$\frac{6435}{128}x^8 - \frac{3003}{32}x^6 + \frac{3465}{64}x^4 - \frac{315}{32}x^2 + \frac{35}{128}$$

Put this sequence command properly into the following plot command so that a plot of all of these functions from $n = 0$ to $n = 8$ appears.

> **plot(.....,x=-1..1);**

Error, `..` unexpected

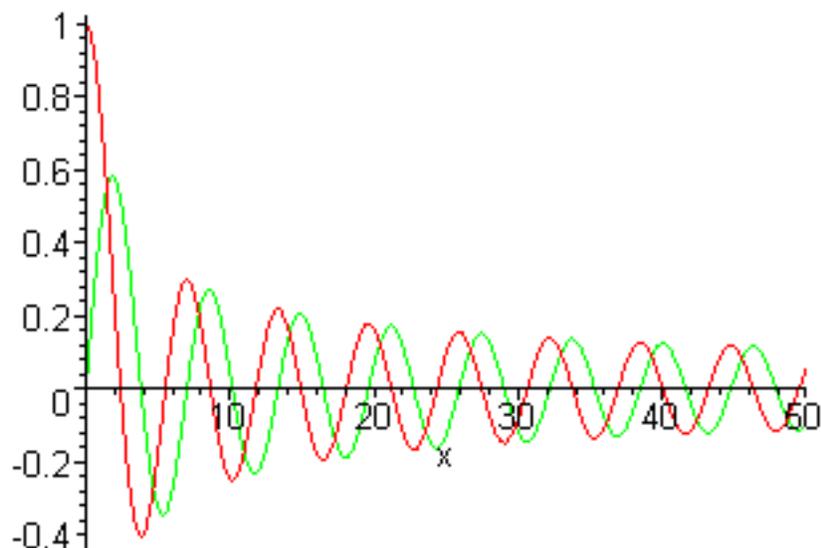
[Go to top of section](#)

Problem 2.14: Bessel functions

Debugging

These functions show up when we do electricity and thermal conduction problems in cylindrical geometry. There are 4 of them that are mostly commonly used: $J_n(x)$, $Y_n(x)$, $I_n(x)$, and $K_n(x)$ and the Maple commands for them are `BesselJ(n,x)`, `BesselY(n,x)`, `BesselI(n,x)`, and `BesselK(n,x)`. The J's and Y's are sort of like cylindrical sines and cosines while the I's and K's are like cylindrical growing and dying exponentials. You also need to know that the Y's and the K's are singular at $x = 0$ so you have to be careful when making plots of them. You can control singularities by choosing the vertical range of the plot to be finite. The Maple command below plots the first two J's. I want you make the following plots: (a) Plot $J_0(x)$ and $Y_0(x)$ on one plot frame from $x = 0$ to $x = 30$. Set **numpoints** large enough that you can see the logarithmic singularity of $Y_0(x)$ near $x = 0$. (b) Plot $J_n(x)$ and $Y_n(x)$ for $n = 1 .. 4$ with the J's on one plot frame and the Y's on another. Plot them from $x = 0$ to $x = 30$. The sequence command **seq** will help you here. (c) Plot $I_n(x)$ for $n = 1 .. 4$ all on one plot frame. Do it from $x = 0$ to $x = 10$. (d) Plot $K_n(x)$ from $n = 1 .. 4$, all on one plot frame. Do it from $x = 0$ to $x = 10$. Take a good look at these functions; they will show up later.

```
> plot([BesselJ(0,x),BesselJ(1,x)],x=0..50);
```



Now use your graphs to answer the following questions.

>

1. (a) The J- and Y-Bessel functions look like sines and cosines after about $x=10$; what is the period of these almost periodic functions? (b) You will notice that their amplitudes are falling off at large x . The

fall off factor is a power law, i.e., $\frac{A}{x^p}$. Find the value of p by plotting $J_0(x)$ and $\frac{A}{x^p}$ on the

same plot and by adjusting A and p until the second function lines up with the top of the Bessel function wiggles.

2. The I- and K-Bessel functions look like exponentials ($e^{(ax)}$) after about $x=10$. Find the

approximate exponential function that fits them out there. You will have trouble doing this because in

addition to the $e^{(ax)}$ -behavior these functions also have the same power-law drop-off factor $\frac{A}{x^p}$

as the J's and Y's. So when you look for the exponential function that comes close to describing what

they do, make your comparison plots between exponentials and $I_n(x) x^p$ and $K_n(x) x^p$. You

will also find it helpful to plot the logarithms of these functions instead of the functions themselves to

handle their huge variation in magnitude. All you want to see is that the logarithms of the $I_n(x) x^p$ functions are parallel to each other and also parallel to the line $y = a x$ for the appropriate choice of a ; then do the same thing for the functions $K_n(x) x^p$.

[Go to top of section](#)

Problem 2.15: The Gamma function

[Debugging](#)

The Gamma function is the analytic continuation onto the real number line of the factorial function for integers. The connection is

$$\Gamma(n + 1) = n! .$$

This function shows up in all kinds of places and it has rather surprising properties for negative values of its argument. All I want you to do is to plot it from -5..5 and verify that it has the correct values at positive integers, and then to marvel that a function that seems so simple on the positive half of the number line could be so messed up on the negative side. Maple's name for it is **GAMMA(x)** ;

Problem 2.16: Complete elliptic integrals

[Debugging](#)

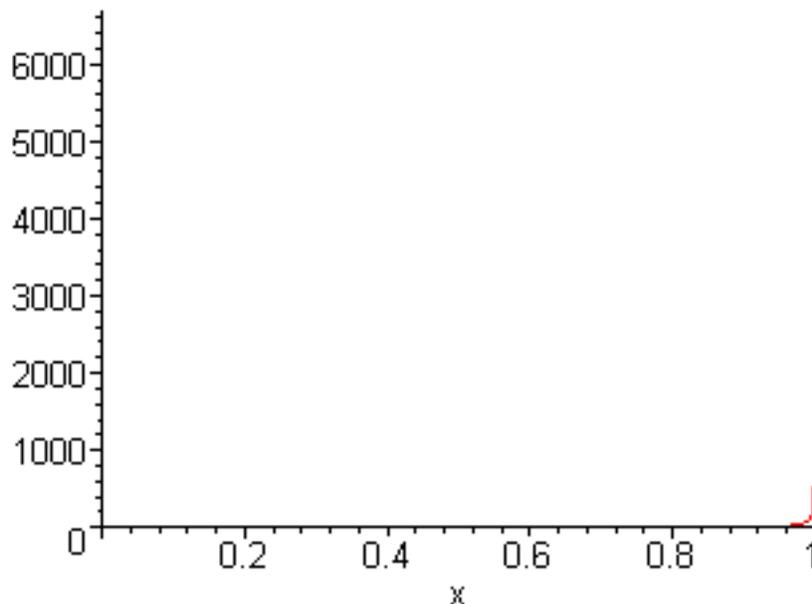
The complete elliptic integral functions $K(k)$ and $E(k)$ are defined by the definite integrals

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin(\phi)^2}} d\phi \quad ; \quad E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin(\phi)^2} d\phi$$

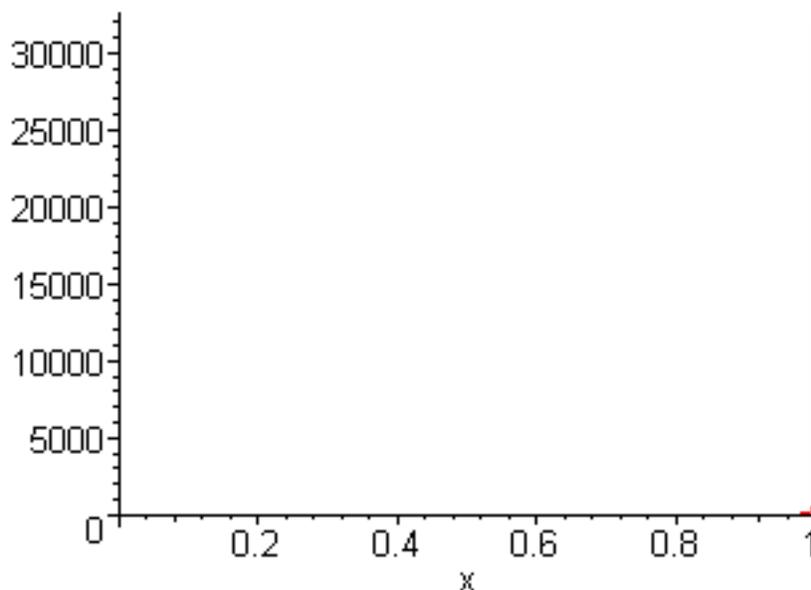
The Maple names for these functions are **EllipticK(k)** and **EllipticE(k)**. These functions are usually used with k in the range 0..1, but they can be analytically continued into the entire complex plane.

Both $K(k)$ and $E(k)$ are $\pi/2$ for $k=0$. At the other end, $k=1$, $K(k)$ is logarithmically singular and $E(k)$ is 1. Plot both functions on the same plot over this range using `numpoints=200` and `1000`; I want you to see how weak a logarithmic singularity is. You should find that almost nothing happens as `numpoints` is changed, except that the vertical plotting range changes a little. To see why this plot behavior indicates a weak singularity, here's what such a pair of plots looks like for a strong singularity:

> **plot(1/(1-x),x=0..1,numpoints=200);**



> **plot(1/(1-x),x=0..1,numpoints=1000);**



>

Problem 2.17: The error function

Debugging

The error function $\text{erf}(x)$ shows up all the time in probability theory and Maple calls it **erf(x)**. If you

have a Gaussian probability distribution, proportional to e^{-x^2} , then the probability that the

variable x lies between $-\infty$ and z is proportional to

$\int_{-\infty}^z e^{-x^2} dx$, which is just what the error function **erf(z)** involves. But for some reason a long

time ago someone decided that the basic function erf should be defined as an integral starting at 0 instead of at $-\infty$, so they defined it this way:

$$\operatorname{erf}(z) = \frac{2 \int_0^z e^{-x^2} dx}{\sqrt{\pi}}$$

. The probability that a Gaussian variable lies between $-\infty$ and z is

given by $\frac{1 + \operatorname{erf}(z)}{2}$. Plot both $\operatorname{erf}(x)$ and $(1 + \operatorname{erf}(x))/2$ in the range $-3..3$.

[Go to top of section](#)

Animations

[Debugging](#)

Perhaps the coolest plots Maple can make are animations. Let's start with two simple wave examples.

The formula for a traveling wave is $y(x, t) = A \sin(kx - \omega t)$. Maple can show you that this

formula travels with the `animate` command. First we will give A , k , and ω numerical values,

then animate the function by plotting many plots in x at successive times, just like a movie.

WARNING: when you execute this group of commands you will see a plot, but it will not be animated; it will just sit there. To make it move you have to click on it and then run it with the player controls that appear on the toolbar. When they come up, take a minute and check out what each one does.

> **restart:with(plots):**

> **A:=1.5;k:=3*Pi/2.;omega:=Pi/2.;**

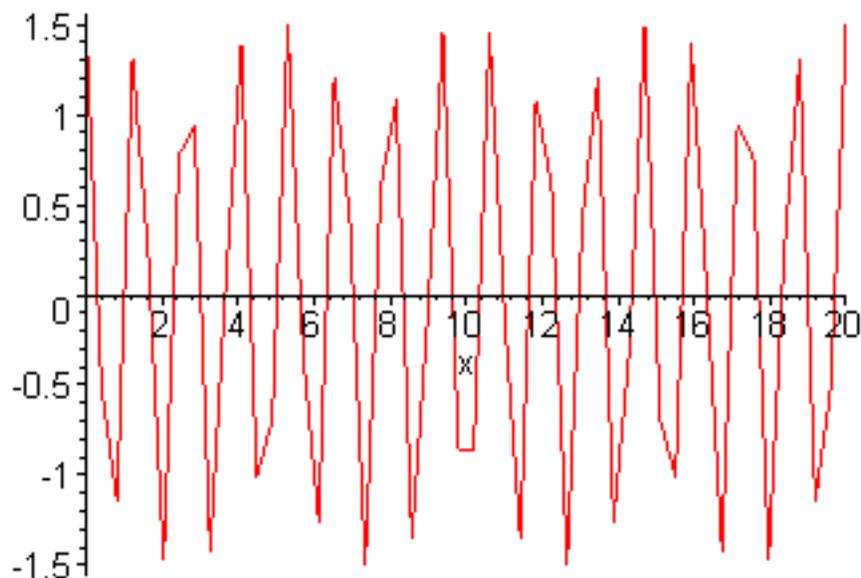
> **animate(A*cos(k*x-omega*t),x=0..20,t=0..20,numpoints=50,frames=10);**

Warning, the name `changecoords` has been redefined

$$A := 1.5$$

$$k := 1.500000000 \pi$$

$$\omega := .5000000000 \pi$$



>

Problem 2.18

Modify the `animate` command above so that the function is smoother in x and so that it runs smoother in time as well by experimenting with the `numpoints` and `frames` settings. Show, using your animation, that the velocity at which the waveltops move (this velocity is called the *phase velocity*) is given by

$v_{\phi} = \frac{\omega}{k}$. Also show that changing the sign of k changes the direction the wave moves.

Problem 2.19

Modify the animation above so that it plots a standing wave instead: $y(x, t) = A \sin(kx) \sin(\omega t)$.

Use a range in x so that the standing wave has 3 zeros between the endpoints and fixed ends as well.

Problem 2.20

Show by making an animation that the sum of two waves of equal amplitude traveling in opposite directions is a standing wave.

Advanced topic: wave packets

In addition to elementary wave properties, animation can also illustrate the more complex behavior of wave packets, including the concepts of group velocity and wave packet diffusion. The way to make a wave packet is to take a whole bunch of waves of the form $A \sin(kx - \omega t)$ with different k 's and add them together, as we did in the section of Fourier analysis when we discussed the uncertainty principle. We can do exactly the same thing here with k , using an amplitude function $A(k)$

centered at k_0 with width Δk to make a wave packet in space with wavelength $\frac{2\pi}{k_0}$ and width

Δx . But the new feature is that this packet moves, and in weird and strange ways. Its behavior is determined by the dispersion relation $\omega = \omega(k)$ of the waves that are being added together. We will start with the simplest case and work our way up to some strange ones that are more than just mathematical curiosities. Every effect that will be shown in this section is physically realized in some kind of electromagnetic wave found in the plasmas that surround the sun and the planets of our solar system.

Suppose the dispersion relation is just the standard non-dispersive one that governs light and sound:

$\omega = kv$, where v is the phase velocity of the waves. In this case every wave of the superposition moves at exactly the same speed, so every part of the wave packet moves at speed v , as shown in the example below.

Define the dispersion relation

> **v:=1;w:=k->k*v;**

$$v := 1$$

$$w := k \rightarrow kv$$

Define the spectrum in k .

> **ko:=1;A:=k-> exp(-(k-ko)^2*40);**

$$k_0 := 1$$

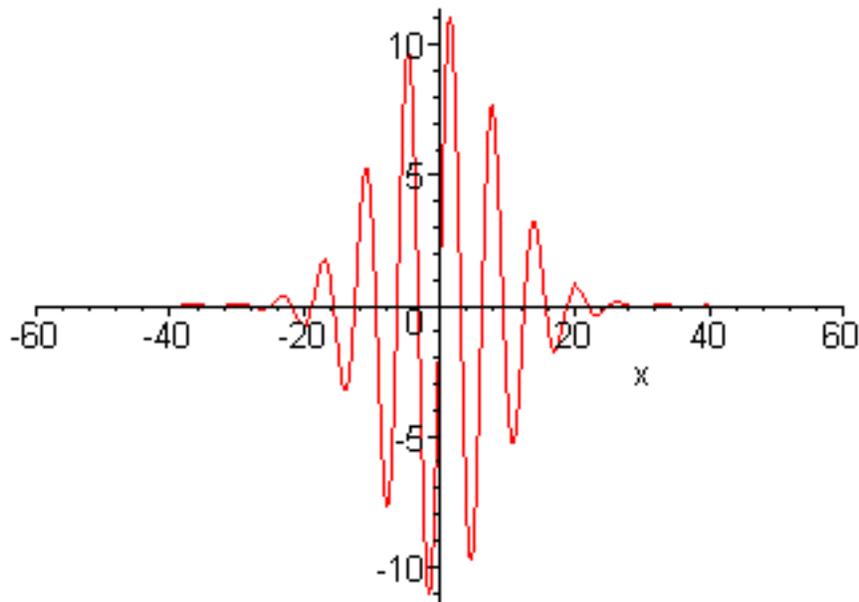
$$A := k \rightarrow e^{(-40(k-k_0)^2)}$$

Now we will look at the wave packet at time t=0.

> **f:=(x,t)-> sum(sin((ko-.5+n/40)*x-w(ko-.5+n/40)*t)*A(ko-.5+n/40),n=0..40);**

$$f := (x, t) \rightarrow \sum_{n=0}^{40} \sin\left(\left(k_0 - .5 + \frac{1}{40}n\right)x - w\left(k_0 - .5 + \frac{1}{40}n\right)t\right) A\left(k_0 - .5 + \frac{1}{40}n\right)$$

> **with(plots):plot(f(x,0),x=-60..60);**



Now we are ready to make it move by using Maple's animation command.

```
> animate(f(x,t),x=-60..60,t=0..180,frames=80,numpoints=400,color=navy);
```

```
>
```

Problem 2.21

If the dispersion relation is not just the linear function $\omega = kv$, then more interesting things can happen. The first of these is that the waves in the packet and the packet as a whole no longer move at the same velocity. Let's try a new dispersion relation using the form

$\omega = kv \left(1 + \frac{ak}{k_0} \right)$. When you study Fourier analysis in more detail you will learn how to show that

while the wave crests move at the phase velocity given by $v_{phase} = \frac{\omega}{k}$, the packet as a whole

moves at the group velocity given by $v_{group} = \frac{d\omega}{dk}$. (a) Find the phase and group velocities for the

dispersion relation given in the introduction to this problem (Maple can help). Choose $k = k_0$ and

obtain v_ϕ and v_g (the standard phase and group velocity notations) as functions of the parameter α . Modify the dispersion relation in the Maple execution groups given in the first part of the problem and run it again with $\alpha = 1$. Try to see the wave crests moving slower than the wave packet as a whole (you will see wavecrests moving backwards through the packet because the group velocity is bigger than the phase velocity). To see this you will have to slow the animation down using the controls on the toolbar. You may also want to put it into continuous play mode and adjust the simulation time so that it runs just long enough that the group velocity covers the distance from $x=0$ to $x=60$. Warning: this takes a while, so be patient. (b) Now for something really weird; choose α so that the group velocity is zero and run the animation again. The packet should sit still while the wave crests move through it.

Problem 2.22

You may have noticed in part (b) of Problem 2.21 that although the packet didn't move much, it did change shape a little. The reason for this is that wave packets can also broaden in time via diffusion, just like a tiny hot spot in a piece of metal spreads out through thermal diffusion. The diffusion coefficient for wave packet broadening is proportional to the magnitude of the second derivative of the dispersion relation (with respect to k). This means that if the dispersion relation has a non-zero second derivative, then the wave packet will spread in time. But to really see it you will need to run the simulation long enough that the effect starts to really matter. This run time is on the order of a few times the absolute value of $1/(w'' \Delta k^2)$ in this problem, where Δk is the width of $A(k)$ (use FWHM). Calculate w'' and verify that it is non-zero, then run the simulation for Problem 2.21 long enough that you can see the spreading.

Problem 2.23

Invent a dispersion relation that has a positive phase velocity and a negative group velocity and verify that it works by modifying and running an animation.

[Go to top of section](#)

Visualizing vector fields

[Debugging](#)

Maple's **fieldplot** command allows you to visualize vector fields in two dimensions. Suppose, for

instance, that you have a magnetic field given by the formulas $B_x = \frac{B_0 y}{a}$ and $B_y = \frac{B_0 x}{a}$. To

visualize the field lines we would do this (setting $B_0 = 1$ and $a = 1$ to make the numbers easy)

> **restart;with(plots):**

> **fieldplot([y,x],x=-1..1,y=-1..1,grid=[20,20],color=red,arrows=SLIM);**

There is also a **fieldplot3d** command which works similarly in three dimensions; you can find out how it works with **?fieldplot3d**.

>

Problem 2.24

(a) Use **fieldplot** to visualize the electric field of a line charge, given by $E = \frac{\lambda (x i + y j)}{2 \pi (x^2 + y^2)}$ where i

and j are the x and y unit vectors. (b) Use **fieldplot** to visualize the magnetic field of a long wire, given

by $B = \frac{\mu_0 I (-y i + x j)}{2 \pi (x^2 + y^2)}$. Experiment with different grid settings and choices for the **arrows** option

(see **?fieldplot**). Note: don't use big numbers in the grid command. If you do, Maple will take forever and when it finishes you will be looking at a solid wall of color without any discernible arrows. I don't usually go above 30x30.

[Go to top of section](#)

3D-surface plots and contour plots[Debugging](#)

To visualize 2-dimensional data in the form of a function $f(x,y)$ Maple gives you two choices: (1) plot

the function values as the height of a surface above the xy plane or (2) draw contours of constant $f(x,y)$ in the xy plane. Which does a better job depends on the problem and on your tastes. Here is an example done both ways.

Consider the function $f(x,y) = e^{(-|x - \sin(y)|)} \left(1 + .2 \cos\left(\frac{x}{2}\right)\right) \left(1 + \frac{.4}{.3 + y^2}\right)$, the

"mountain range" function. (Don't worry about where this function came from. I just made it up because it looks cool, like a mountain range.) A three-dimensional surface plot is done with **plot3d** like this

```
> restart;
> f:=exp(-abs(x-sin(y)))*(1+.2*cos(x/2))*(1+.4/(.3+y^2));
> plot3d(f,x=-6..6,y=-6..6);
```

This looks rather interesting, but the detail is too crude to really see what is going on, and we might want to use a different viewing angle too. These things are controlled with options which you can read about in **?plot3d[options]** or you can just click on the figure and move the cursor while holding the left button down to cause the view to rotate. If you go into the Options menu on the toolbar and change the plot display from inline to window you will get a figure with its own set of tools that will let you change lots of things about the way it looks.

Here is an improved plot using options in the command line:

```
> plot3d(f,x=-6..6,y=-6..6,grid=[60,60],labels=["x","y","f"],orientation=[45,30],axes=boxed,
shading=xy,view=0..3);
```

```
>
```

As you can see the surface grid is controlled by the **grid=[m,n]** option, the labels on the axes are controlled by the **labels** option, and the viewing angle is controlled by orientation. A word of warning: the mathematicians who wrote Maple are used to having ϕ be the angle in spherical coordinates that comes down from the z-axis and θ be the azimuthal angle that winds around the z-axis in the xy plane. *This is exactly backwards from the way we do it in physics.* So to be consistent with the way we do spherical coordinates in physics, the form of the orientation option is **orientation=[ϕ , θ]** which is exactly backwards from what you will read in online help. In addition, the angle ϕ indicates the direction along which the viewer is sighting rather than the angular position of the viewer. This is weird

enough that you will just have to experiment until you get what you want. The vertical size of the plotting box is controlled by the **view=zmin..zmax** option. Finally, the appearance of the plot is altered by the shading command which can be **shading=z**, **shading=xy**, **shading=xyz**, or a few other things found under **?plot3d** . Experiment and see what looks best.

You can look at the same mountain with a contour, or topographic map, view like this (note that to use this command you have to to **with(plots)** first)

```
> restart;
> with(plots):
> f:=exp(-abs(x-sin(y)))*(1+.2*cos(x/2))*(1 + .4/(.3+y^2));
> contourplot(f,x=-6..6,y=-6..6,contours=40,grid=[30,30],coloring=[green,blue]);
```

Note that the **coloring=[green,blue]** plots the low values of the function in green, the high values in blue, and in-between values in colors between green and blue. You can use any two Maple colors you want in this option (see **?plot[colors]**)

```
> ?plot[colors]
```

Look more carefully at this contour plot and compare it with the 3d mountain picture--in Maple 6 they don't look like the same objects. This is a bug in the original release of Maple 6. But if you have Maple 6.01, or later, this bug has been fixed and contour plotting is done right.

```
>
```

[Go to top of section](#)

[Go to top of chapter](#)

```
>
```

```
>
```

[Chapter3.mws](#)

- [Limits](#)
 - [Differentiation](#)
 - [Integration](#)
 - [Elementary integrals](#)
 - [Integrating special functions](#)
 - [Multiple Integrals](#)
 - [Series expansions](#)
 - [Sums](#)
-

Chapter 3: Calculus

One of the things Maple can help you with the most is calculus. It can take derivatives, do integrals, make series expansions, do infinite sums, and a whole lot more. In this chapter we will concentrate just on the basics.

Limits

[Debugging](#)

The most basic idea in calculus is that of a limit, and Maple knows how to do them. For example, to find

the value of the limit $\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$, just use the Maple command **limit** on the expression like

this

```
> limit(sin(3*x)/x,x=0);
```

3

You can also do it with a Maple function

```
> y:=x->sin(3*x)/x;limit(y(x),x=0);
```

$$y := x \rightarrow \frac{\sin(3x)}{x}$$

3

You can use **?limit** to see some details about how this command works, but this is about all there is to it.

>

Problem 3.1

Try this one just for practice $\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x^2}$.

[Go to top of section](#)

Differentiation

[Debugging](#)

Derivatives are relatively simple, so this section is simple. Maple can take derivatives of elementary functions and special functions with equal ease, so there is almost nothing to do in this section except show you Maple's two differentiation commands, one for expressions and one for functions.

First let's do expressions. I suggest you use the following forms for taking the 1st, 2nd, and third derivatives, as illustrated below using the tangent function. You can either use **diff**, which gives you the derivative directly, or **Diff** and **value**, which display the derivative to be taken, then evaluate it. **Diff** is actually more useful than you might think because it gives you a chance to see if the derivative you have asked Maple to take is the one you really want.

> **diff(tan(x),x);**

$$1 + \tan(x)^2$$

> **diff(tan(x),x\$2);**

$$2 \tan(x) (1 + \tan(x)^2)$$

> **d:=Diff(tan(x),x\$3);**

> **d:=value(d);**

$$d := \frac{\partial^3}{\partial x^3} \tan(x)$$

$$d := 2 (1 + \tan(x)^2)^2 + 4 \tan(x)^2 (1 + \tan(x)^2)$$

> **d:=simplify(d);**

$$d := 2 + 8 \tan(x)^2 + 6 \tan(x)^4$$

Now let's see how to do functions.

> **f:=x->tan(x)/x;**

$$f := x \rightarrow \frac{\tan(x)}{x}$$

Diff doesn't work on functions; instead we have to use Maple's **D** command. This is a very powerful command in spite of its small size. It can take multiple derivatives of multiple functions (to see how to use all of its features type **?D**) but we will only be taking first derivatives of a single function. The first derivative is easy

> **fp:=D(f);**

$$fp := x \rightarrow \frac{1 + \tan(x)^2}{x} - \frac{\tan(x)}{x^2}$$

Note that assigning the result of **D(f)** to fp produces the function fp(x). There is more than one way to

take higher derivatives, but here is the most general one.

> **fpp:=D[1\$2](f);**

$$fpp := x \rightarrow 2 \frac{\tan(x) (1 + \tan(x)^2)}{x} - \frac{2 (1 + \tan(x)^2)}{x^2} + \frac{2 \tan(x)}{x^3}$$

In the square brackets the 1 means differentiate with respect to the first (in this case the only) variable in the argument list and \$2 means do it twice just as in **diff**.

>

Well, that's about it. Here are some exercises to practice on.

Problem 3.2

Take the indicated derivatives of the functions listed below. Use expression form on most of them but use function form on (a) and (d). When you get a mess try using **simplify** to clean it up. You will discover that **simplify** doesn't work on functions; to make the result look nice use the mouse to copy the mess you want to simplify to the clipboard, assign it to a new variable, cut off the extraneous junk, execute the command, then rebuild the derivative function using cut and paste again. This combination of Maple and editing is a good way to do mistake-free algebra.

$$(a) \frac{\partial^3}{\partial x^3} \sqrt{1+x^3} \quad (b) \frac{\partial}{\partial x} J_0(x) \quad (c) \frac{\partial}{\partial x} I_1(x) \quad (d) \frac{\partial^2}{\partial x^2} e^{\tan(x)} \quad (e) \frac{\partial}{\partial x} \Gamma(x)$$

$$(f) \frac{\partial}{\partial x} \operatorname{erf}(x) \quad (g) \frac{\partial}{\partial k} K(k) \quad (\text{this is the complete elliptic integral of the first kind, Maple's$$

EllipticK.)

Problem 3.3

Here's a find-the-maxima-and-minima problem like ones you used to do in high school. Consider the

function $\ln(x) J_0(x)$. (Note: I am using the word function in the mathematical sense, not in the Maple sense. This problem is easier if you just use a Maple expression to define the function above.) (a) First plot this function on the interval $[0,10]$. (b) Look at the graph and find, approximately, the values of x where the function has maxima and minima. Then take the derivative of the function and use the **fsolve** command to find these values of x precisely. If your expression for the derivative is called f , and if you wanted to find a zero of it near 1.1, you would do this: **fsolve(f,x=1.1)**;

Problem 3.4

In quantum mechanics you will encounter a close relative of the Legendre functions $P_n(x)$ that we have seen already. These new functions are called the Associated Legendre functions P_n^m . For each integer n these functions are defined for values of m in the interval $[0..n]$, with the $m=0$ function being plain old $P_n(x)$. These functions are defined in terms of derivatives of the Legendre functions:

$$P_n^m = (-1)^m (1-x^2)^{\binom{m}{2}} \text{diff}(P_n(x), x \$ m)$$

. This definition is too cumbersome for most

computer languages, but Maple handles it with ease because it thinks symbolically instead of numerically. Here is a function that evaluates it

> **with(orthopoly);**

[G, H, L, P, T, U]

> **Pnm:=(n,m,x)-> (-1)^m*(1-x^2)^(m/2)*diff(P(n,x),x\$m);**

$$Pnm := (n, m, x) \rightarrow (-1)^m (1 - x^2)^{\binom{1}{2}m} \text{diff}(P(n, x), x \$ m)$$

Before we do anything fancy with it we ought to test it, so let's put in numbers for n , m , and x .

```
> Pnm(3,1,.5);
```

```
Error, (in Pnm) wrong number (or type) of parameters in function diff
```

Well, we're in trouble again. The problem is with what **P(n,x)** returns. As we saw in the section in Chapter 2 on this function, it doesn't return a number; it returns a polynomial. When we put in 0.5 for x it goes into our Pnm function, defined above, in place of x , and then **diff** tries to take a derivative with respect to 0.5, which makes no sense. Watch what happens when we put in a variable in place of x instead of a number

```
> Pnm(3,1,t);
```

$$-\sqrt{1-t^2} \left(\frac{15}{2}t^2 - \frac{3}{2} \right)$$

So if you want a numerical result you have to do something like this

```
> a:=Pnm(3,1,t);t:=0.5;a;
```

$$a := -\sqrt{1-t^2} \left(\frac{15}{2}t^2 - \frac{3}{2} \right)$$

$$t := .5$$

$$-.3247595264$$

This is annoying, but on the other hand, just think about it; why do you need a number in Maple anyway? You want to plot the function, differentiate it, integrate it, use it in differential equations, etc..

What could be better than having an explicit expression for the thing? So Maple doesn't think this is a problem; it's a feature! And this feature is shared by all of the orthogonal functions you get when you use **with(orthopoly)**.

There is one other annoying thing about the function Pnm. Watch what happens when we try to evaluate it with m=0

```
> Pnm(5,0,x);
```

```
Error, (in Pnm) wrong number (or type) of parameters in function diff
```

With m=0 it is supposed to give us Pn(x) back, but it doesn't. The reason it doesn't is that we just asked it to take the 0th derivative of a function, which Maple's **diff** command can't handle. Later when we study procedures we will come back to this problem and fix it so it works with m=0.

```
>
```

Ok, I have shown you how this can be done. Now I want you to make a plot of the 5 associated Legendre functions that go with P(5,x), i.e., n=5 and m=1,2,3,4,5. Plot them from x=-1..1. Put all five plots on one set of axes with different colors so you can see what happens as m ranges from 1 up to n=5. After you look at the picture you will probably want to rescale the functions so they all look about the same size. In the upcoming section on integration we will redo this plot and learn a natural way to make the functions nearer to the same size.

Problem 3.5

Here is an electricity problem borrowed from the integration section that follows this one. The potential function for a charged hemisphere of radius R with surface charge density σ as a function of z, where z goes up the symmetry axis of the hemisphere, is given by the expression

```
> V:=-1/2*sigma*R*(-sqrt(R^2+z^2)+sqrt((z-R)^2))/(z*e0);
```

$$V := -\frac{1}{2} \frac{\sigma R (-\sqrt{R^2 + z^2} + \sqrt{(z - R)^2})}{z e_0}$$

where ϵ_0 is shorthand for the electrical constant ϵ_0 .

>

The electric field component E_z can be obtained from the potential V by differentiating:

$E_z = -\left(\frac{\partial}{\partial z} V\right)$. Use Maple to take this derivative and obtain an (ugly) expression for E_z .

Simplify it. You will see an unfamiliar function called **csgn**; look this function up with **?csgn** and make sure you understand what it does. Then set $\sigma = 1$, $R = 1$, and $\epsilon_0 = 1$ and plot both V and

E_z from $z = -4$ to $z = 4$. It is a theorem in electromagnetism that when you cross surface

charge density, the electric field jumps up by $\frac{\sigma}{\epsilon_0}$. (You will notice that in the definition of V above I

used ϵ_0 in place of ϵ_0 . This was intentional. Avoid subscripted variables whenever possible because subscripts in Maple reference array elements.) Verify on your plot that the correct jump is obtained. On your graph negative z is below the circular rim of the hemisphere; positive z from 0 to R is inside the hemisphere; and positive z from R to infinity is above the dome. Interpret your graph physically and convince yourself that it makes sense.

Problem 3.6

Here is a fancy kind of differentiation called *implicit differentiation* that Maple knows how to do.

Suppose that you have an equation involving both x and y , like this one $x^2 + y^3 = 3$. You

would like to find $\frac{dy}{dx}$ without solving for $y(x)$. The way to do this is to differentiate this equation

implicitly to get $2x + 3y^2 \left(\frac{\partial}{\partial x} y\right) = 0$, then solve for $\frac{dy}{dx}$. Maple knows how to do it, provided

that you tell it that y depends on x , like this.

> **restart;**

> **eq:=x^2 + y(x)^3 = 3;**

$$eq := x^2 + y(x)^3 = 3$$

> **deq:=diff(eq,x);**

$$deq := 2x + 3y(x)^2 \left(\frac{\partial}{\partial x} y(x) \right) = 0$$

> **dydx:=solve(deq,diff(y(x),x));**

$$dydx := -\frac{2}{3} \frac{x}{y(x)^2}$$

If you don't want to type $y(x)$ all of the time, you can use Maple's **alias** command to tell it to turn y into $y(x)$ (only in Maple's internal processing) whenever it is encountered.

>

> **restart;**

Allow us to use y in place of $y(x)$

> **alias(y=y(x));**

y

> **eq:=x^2 + y^3 = 3;**

$$eq := x^2 + y^3 = 3$$

> **deq:=diff(eq,x);**

$$deq := 2x + 3y^2 \left(\frac{\partial}{\partial x} y \right) = 0$$

> **dydx:=solve(deq,diff(y,x));**

$$dydx := -\frac{2x}{3y^2}$$

>

Here's an example where this shows up in physics. The dispersion relation for electromagnetic waves in a plasma is $\omega^2 = \omega_p^2 + k^2 c^2$, where ω_p is a frequency called the plasma frequency. The

phase velocity of these waves is given by $\frac{\omega}{k}$ while the group velocity is given by $\frac{d\omega}{dk}$. First use

Maple to find formulas for both the phase and group velocities in terms of ω_p , k , and c by solving for $\omega(k)$ and differentiating. Then use implicit differentiation to get the group velocity in terms of k , c , and ω .

Finally, Maple also knows how to take partial derivatives. Consider the function of x and y

$f(x,y) = \frac{\cos(xy)}{y}$. Here are its derivatives with respect to x , y , and both x and y , using the

expression form

> **restart;f:=cos(x*y)/y;**

$$f := \frac{\cos(xy)}{y}$$

> **diff(f,x);diff(f,y);diff(f,x,y);**

$$-\sin(xy)$$

$$-\frac{\sin(xy)x}{y} - \frac{\cos(xy)}{y^2}$$

$$-\cos(xy)x$$

And here is the same thing using Maple function notation

> **restart;f:=(x,y)->cos(x*y)/y;**

$$f := (x, y) \rightarrow \frac{\cos(xy)}{y}$$

> **D[1](f);D[2](f);D[1,2](f);**

$$(x, y) \rightarrow -\sin(xy)$$

$$(x, y) \rightarrow -\frac{\sin(xy)x}{y} - \frac{\cos(xy)}{y^2}$$

$$(x, y) \rightarrow -\cos(xy)x$$

>

Problem 3.7

Find the two first derivatives and all three second derivatives (double x, double y, and mixed xy) of the following function

$K\left(\sqrt{\frac{4xy}{(x+y)^2}}\right)$, where K is the complete elliptic integral **EllipticK**. Use expression notation with **diff**. Try using **expand** and **simplify** to clean up the messes that result.

[Go to top of section](#)

Integration

The single thing you will use Maple for the most is integration. In fact, many of you don't have any more idea of what an integral table is than you have about slide rules. Mostly this is OK because Maple is easily available to you and is pretty good. But it doesn't know how to do everything (as you will see in the some of the examples in this section), so you need to know where to go when Maple fails. The best place to go is a math reference book called *A Table of Series and Integrals* by Gradshteyn and Ryzhik. You can find it in the math reference section of the library, or maybe even in our department library, if some faculty member hasn't stolen it.

Elementary integrals

[Debugging](#)

Maple knows how to do all of the integration problems you encountered in your first calculus course. The command to do this is **int**, and with expressions you use it like this

```
> int(sin(x),x);
```

```
-cos(x)
```

or

```
> f:=sin(x)*x;int(f,x);
```

```
f:= sin(x) x
```

```
sin(x) - x cos(x)
```

Note: do not use $f(x)$ as the argument if f is an expression.

With functions the integration command works like this

> **g:=(x,y)->sin(x*y)*x;**

$$g := (x, y) \rightarrow \sin(x y) x$$

> **int(g(x,y),x);**

$$\frac{\sin(x y) - x y \cos(x y)}{y^2}$$

There is also an inert form of **int**, namely **Int**, which is used to display integrals. You would use this form for documenting a worksheet. Try this:

> **s1:=Int(exp(x),x);**

$$s1 := \int e^x dx$$

But be warned: **Int** only displays, it doesn't do math. "But if it doesn't do anything, why would I want to use it?", you ask? Because it will help you see if you have entered the integral properly, which makes **Int** a very valuable debugging tool. After you have seen that it looks right in displayed form, use **value** (**s1**) to get the answer. So the right way to do the simple integral above and get an answer is this

> **s1:=Int(exp(x),x);**

> **s1:=value(s1);**

$$s1 := \int e^x dx$$

$$s1 := e^x$$

I suggest that you always do integrals this way, combining **Int** and **value**. It is one of those habits that will decrease the number of hours you will spend looking for stupid mistakes.

And you can do definite integrals too, like this

```
> s2:=Int(tan(x),x=0..1);
```

```
> s2:=value(s2);
```

$$s2 := \int_0^1 \tan(x) dx$$

$$s2 := -\ln(\cos(1))$$

And if you need a number there is always

```
> evalf(s2);
```

```
.6156264703
```

Oh, and if you just want to get a numerical answer without going through **evalf**, just give **int** floating point limits and you will get it right away.

```
> s2:=Int(tan(x),x=0..1.);
```

```
> value(s2);
```

$$s2 := \int_0^{1.} \tan(x) dx$$

```
.6156264704
```

You should also know that Maple can do integrals where the limits are infinite, but you may need to give it some guidance by using the **assume** command. Well, that's about all you have to know if Maple can do the integral. Try **?int** to see some of the extra integration options Maple has available. Now let's do some exercises.

```
>
```

Problem 3.8

Do the following integrals with Maple using expression notation for (a)-(d) and function notation for (e)-(g). Obtain numerical values for (e) and (f). You will have trouble with (g), and even when you get it to work the answer will look ugly. Try simplifying it with **simplify** .

$$(a) \int \ln(x) dx \quad (b) \int \sqrt{1-x^2} dx \quad (c) \int \frac{x}{1+x^3} dx \quad (d) \int \cosh(x) dx$$

$$(e) \int_0^1 \sqrt{\frac{1+x}{1-x}} dx \quad (f) \int_0^{\frac{1}{2}} \frac{x}{x^3-1} dx \quad (\text{try using both } 1/2 \text{ and } 1./2. \text{ as the upper limit}) \quad (g)$$

$$\int_0^{\infty} e^{(-a x)} \cos(x) dx \quad (\text{Don't know how to handle } \infty \quad ? \text{ Use online help.})$$

Problem 3.9

A common way that integrals show up in physics problems is in calculating voltages and electric fields (remember sophomore electricity and magnetism?) Here is one that you might encounter in Physics 441 that Maple knows how to do. Consider a hemisphere described in spherical coordinates by $r = R$ and

$\theta = 0 \dots \frac{\pi}{2}$. Let it have surface charge density σ . You are to find the electrostatic potential

(voltage) $V(z)$ along the z -axis.

I will get you started. The vector distance to the observation point on the z -axis is $\mathbf{r} = z \mathbf{k}$, where \mathbf{k} is

the unit vector in the z -direction. The vector distance to a little bit of charge

$dq = \sigma R^2 \sin(\theta) d\theta d\phi$ is $\mathbf{r}' = R \mathbf{r}'$ where \mathbf{r}' is a unit vector pointing from the origin to the little

bit of charge. Using the law of cosines we can then write

$$V(z) = \frac{\iint \frac{\sigma R^2 \sin(\theta)}{\sqrt{R^2 + z^2 - 2Rz \cos(\theta)}} d\theta d\phi}{4\pi\epsilon_0}$$

where the integral sign here means integrate over both θ and ϕ . The ϕ integral just gives

2π and we are left with

$$V(z) = \frac{\int_0^{\pi} \frac{\sigma R^2 \sin(\theta)}{\sqrt{R^2 + z^2 - 2Rz \cos(\theta)}} d\theta}{2\epsilon_0}$$

Use Maple's **Int** command to build a function $V(z)$ that we can use later. The syntax for function building is this (setting $V(z)$ to the dying exponential function).

> **V:=z->exp(-z);**

$$V := z \rightarrow e^{(-z)}$$

Note: recall that you had better not use the symbol $V(z)$; Maple interprets this literally, i.e., $V(z)$ is the name of the function you are defining, requiring you to use the awkward syntax $V(z)(z)$ when using it. As defined above you type $V(z)$ when you use it. You can do the same thing with **int**, like this **V:=z->int(...)**. In place of the symbol ϵ_0 use $e0$, and define the constants so Maple can plot numbers.

Here, I'll get you started; you fill in what goes in (...)

> **restart:e0:=8.854e-12;sigma:=1e-10;R:=0.5;V:=z->int(sigma*R^2*sin(theta)/sqrt(R^2+z^2-2*R*z*cos(theta)),theta = 0 .. Pi/2)/(2*e0);**

$$e0 := .8854 \cdot 10^{-11}$$

$$\sigma := .1 \cdot 10^{-9}$$

$$R := .5$$

$$V := z \rightarrow \frac{1}{2} \int_0^{\frac{1}{2}\pi} \frac{\sigma R^2 \sin(\theta)}{\sqrt{R^2 + z^2 - 2 R z \cos(\theta)}} d\theta$$

$e0$

You will notice that Maple doesn't give you a formula for the answer this time; it just displays the integral. I don't know why. But now that it is a function you can plot it as a function of z . Do so, from $z=-5$ to $z=5$.

Now calculate the total charge on the hemisphere and overlay plots of the true potential and the

corresponding point charge potential. The two plots should agree for $|z| \gg R$. You can get a better fit by putting the point charge somewhere other than at the origin; can you graphically find about the best value of z at which to put the point charge? Recall that the formula for the potential of a point charge is

$$V(r) = \frac{q}{4 \pi \epsilon_0 r} .$$

>

Problem 3.10

Another place integrals show up is in classical mechanics. Consider the equation of motion of a pendulum:

$$\frac{\partial^2}{\partial t^2} \theta = -\Omega^2 \sin(\theta)$$

where Ω is the small-oscillation frequency of the pendulum. If we multiply this equation by $\frac{d\theta}{dt}$

and integrate once with respect to time, we get the conservation of energy theorem for this problem:

$$\frac{\left(\frac{d\theta}{dt}\right)^2}{2} = \Omega^2 \cos(\theta) - \Omega^2 \cos(\theta_0)$$

where θ_0 is the initial angular position of the pendulum before we release it from rest. We would like to know how long it takes the pendulum to get from its initial angle down to $\theta = 0$ because this time

is $\frac{T}{4}$, one-quarter of the period of a pendulum with initial amplitude θ_0 . (a) Show that the energy equation above can be rewritten in the form

$$\frac{d\theta}{\Omega \sqrt{2(\cos(\theta) - \cos(\theta_0))}} = dt.$$

This means that the quarter period of the motion is given by the integral

$$\int_0^{\theta_0} \frac{1}{\Omega \sqrt{2(\cos(\theta) - \cos(\theta_0))}} d\theta = \frac{T}{4}.$$

(b) Have Maple do this integral in an attempt to get a formula for the period T of a pendulum, using s and $s0$ instead of θ and θ_0 . Once you try it you will probably be unimpressed with the beauty of the result. Part of the problem is that Maple doesn't know what you are going to use for θ_0 , so it gives you something incredibly general (as far as it knows, you might even want θ_0 to be complex). To tie Maple down a little, use the **assume** command, like this

```
> assume(theta0,real,0<theta0,theta0<Pi);
```

(Note: when you see these variables again they will have little \sim signs attached to them. This is Maple's way of showing that they have been restricted in meaning by an **assume** command.) Now you can try integrating again; it will look a little better, but it is still ugly.

```
>
```

You can often help Maple out by using integration tricks, like u-substitutions. Try this integral again, but

use the identity $\cos(\theta) = 1 - 2 \sin\left(\frac{\theta}{2}\right)^2$ to change the functions inside the square root. You and

Maple are a team; don't be afraid to use what you have learned in your math classes.

(c) Plot the period T as function of initial angle θ_0 from 0 to π . Does your plot make physical sense? Check the plot for θ_0 both small and near π .

Problem 3.11

Integrals also show up in physics when calculating the moments of inertia of spinning objects. The

moment of inertia is defined by the integral $\int s^2 \rho dV$ where s is the perpendicular distance from

the spin axis to a point in the object, where ρ is the mass density, and where dV is the differential

volume element, e.g., $dx dy dz$ in Cartesian coordinates. (a) Use Maple to find the moment of inertia

of a sphere of radius R about an axis through its center. This is most easily done in spherical

coordinates in which $dV = r^2 \sin(\theta) dr d\theta d\phi$. If we use the z axis as the spin axis, then

$s = r \sin(\theta)$ (draw a picture and convince yourself that this is true). Assume that the mass density is

constant so that $\rho = \frac{M}{\frac{4\pi R^3}{3}}$. The answer is $\frac{2MR^2}{5}$. (b) Use Maple to find the moment of

inertia of a cone of height L and base radius R about the symmetry axis of the cone (the axis from

the sharp tip to the center of the bottom). This is most easily done in cylindrical coordinates in which

$dV = r dr d\phi dz$ and $s = r$. Assume again that the mass density is constant so that

$$\rho = \frac{M}{\frac{\pi R^2 L}{3}} .$$

[Go to top of section](#)

Integrating special functions

Debugging

Maple can also do integrals of special functions like Bessel functions, Legendre functions, etc.. Sometimes it will give you a nice answer and sometimes it will either just give you the integral back (meaning it doesn't know how to do it) or give you an answer so ugly it makes your eyeballs bleed. When either of these bad things happens, don't give up. Sometimes if you play with the integral a little you can get a better answer. And if all you need is a number, then Maple can always give you one via **evalf**. For instance, suppose you needed to do this Bessel function integral

$$\int_0^1 J_0(x) dx$$

You might first try this

```
> restart;
> s1:=Int(BesselJ(0,x),x=0..1);
> value(s1);
```

$$s1 := \int_0^1 \text{BesselJ}(0, x) dx$$

BesselJ(0, 1) –

$$\frac{1}{2} \pi (\text{BesselJ}(0, 1) \text{StruveH}(1, 1) - \text{BesselJ}(1, 1) \text{StruveH}(0, 1))$$

and be annoyed because you haven't ever heard of Struve functions. What you really want is just a number, which you can get via **evalf** (Maple knows all about Struve functions even if you don't) or by the following subtle change in the command (watch the decimal points).

> **s1:=Int(BesselJ(0,x),x=0..1.);**

> **value(s1);**

$$s1 := \int_0^{1.} \text{BesselJ}(0, x) dx$$

.9197304101

That little decimal point on the 1. makes all the difference. Ok, here are some exercises for you to practice on.

>

Problem 3.12

Try doing the following Bessel function integrals both at indefinite integrals and as definite integrals on the interval [1.0..2.0].

$$(a) \int J_0(x) dx \quad (b) \int J_0(x) x dx \quad (c) \int K_0(x) dx \quad (d) \int I_1(x) dx$$

(Note that you get these functions with **BesselJ(0,x)** , **BesselK(0,x)** , and **BesselI(1,x)** .

Problem 3.13

Try doing the following Legendre polynomial integrals both as indefinite integrals and as definite integrals on the interval [-1.0..1.0].

$$(a) \int P_2(x) x^2 dx \quad (b) \int P_3(x) P_5(x) dx \quad (\text{Note: to get these functions you have to use the}$$

command **with(orthopoly)** first, then **P(3,x)**, etc..)

Problem 3.14

You might have been surprised at the answer to part (b) of the previous problem when you did it as a definite integral. This is not just a surprise, it is a miracle, and it goes by the name of orthogonality. You will study this idea in detail in Physics 318, but this seems like a nice place to give you an introduction to it.

If I were to ask you to check whether two vectors were orthogonal, you would know just what to do. You would take their dot product, and if it turned out to be zero, then you would know that they are orthogonal. And how would you take the dot product? You would multiply the like components of the two vectors together and add them up, e.g.,

$$[1,2,3] \text{ dot } [4,5,6] = 4 + 10 + 18 = 32$$

Well, an integral like this $\int_{-1}^1 P_3(x) P_5(x) dx$ is really just the same thing. At each value of x

the two function values are multiplied together and the integral adds them all up. So it makes sense to say that two functions are orthogonal. It turns out that many functions have nice orthogonality properties, and in this exercise you will use Maple to see what they are like.

(a) Show, using Maple, that the functions $\sin(mx)$ and $\cos(nx)$ are orthogonal to each other when integrated over the interval $[0, 2\pi]$ by finding the values of the following integrals both with m and n unequal and with $m = n$.

$$\int_0^{2\pi} \cos(mx) \cos(nx) dx \quad \int_0^{2\pi} \sin(mx) \sin(nx) dx \quad \int_0^{2\pi} \cos(mx) \sin(nx) dx$$

(Hint: you will get a mess unless you use **assume** to tell Maple that m and n are integers.

Apparently Maple assumes more than that they are integers; it also assumes that they are different integers, so you will have to do the $m = n$ case by using the command **m:=n**.)

(b) Show, using Maple (this problem is an extension of Problem 3.13 above), that the Legendre functions $P_n(x)$ are orthogonal to each other when two functions with different n -values are

integrated over the interval $[-1.0..1.0]$. Also find the value of $\int_{-1}^1 P_n(x)^2 dx$. You will need to use

the **with(orthopoly)** command to tell Maple you are going to use these functions. And when you try to find the value of the integral of $P_n(x)^2$, you will probably need to try several different values of n and see if you can find the general formula empirically. This will be easier if you don't force floating point answers by using decimal points on the limits, i.e., use 1 and -1 instead of -1. and 1..

(c) Show, using Maple, that the Bessel functions $J_0(\alpha_n x)$ are orthogonal to each other with respect

to the integral $\int_0^1 J_0(\alpha_p x) J_0(\alpha_q x) x dx$, where the numbers α_p and α_q are the zeros of

$J_0(x)$. These zeros are available from the Maple function **BesselJZeros(n,m)**, where n is the order of the Bessel function, i.e., $n = 0$ for J_0 , $n = 1$ for J_1 , etc., and where m counts

the zeros (p and q are m -values), starting with the first one. (For example **BesselJZeros(0,1)** gives 2.404825558; notice the decimal point--it is important.) Then see if you can find a formula for this integral when $p = q$. (Hint: when you set $p = q$ you will just get meaningless numbers because **BesselJZeros** gives you floating point numbers back. You can do better than this. Replace **BesselJZeros**

(0.,n) with **BesselJZeros(0,n)** (no decimal point) and let Maple try to do the integral symbolically. You will get a simple answer.

Problem 3.15

Let's go back to the associated Legendre functions and plot them so that they all have about the same size. We could scale them by their maximum values, but then we would have to find the zeros of large polynomials, which is hard. It is actually easier to "calculate their magnitudes" in the same function-dot-product sense used in Problem 3.14. To find the magnitude of a vector we square the components, add the squares up, then take the square root. The corresponding operation for functions is: square the function, integrate it over the relevant interval, then take the square root. Here is the associated Legendre function definition again from Problem 3.4

> **Pnm:=(n,m,x)-> (-1)^m*sqrt(1-x^2)^m*diff(P(n,x),x\$m);**

$$P_{nm} := (n, m, x) \rightarrow (-1)^m \sqrt{1-x^2}^m \operatorname{diff}(P(n, x), x \$ m)$$

The magnitude of this function is obtained from $|P_{nm}| = \sqrt{\int_{-1}^1 P_{nm}(x)^2 dx}$. Redo the plots

in Problem 3.4, but divide each of the plotted functions by this magnitude and check to see that they all have about the same size on the new plot.

>

Problem 3.16

Finally, let's see what Maple does with the integrals of a few miscellaneous functions. Try to get Maple to do (a-c) both as indefinite integrals and as definite integrals between .5 and 1. Do (d) only as a definite integral.

(a) $\int \Gamma(x) dx$

(b) $\int \operatorname{erf}(x) dx$

(c) $\int K(x) dx$ (Maple's **EllipticK**)

(d) $\int_0^1 e^{(-x^{2.3})} dx$ (Note: try this integral with both integer 1 and floating point 1. as the upper

limit.)

This should teach you that sometimes Maple can't do indefinite integrals, that it can give you a floating point number for a definite integral but you might have to insist on it by using **evalf**, and that sometimes

it will *give you the wrong answer!* (Plot $e^{(-x^{2.3})}$ between 0 and 1 to figure out which of the two

answers Maple gave you for (d) is wrong.) Maple doesn't make many mistakes, but occasionally it louses up. This means that you have to be very careful and not trust Maple, or any other computer program, too much. Indeed, a good rule to follow upon receiving an answer from any computer program is to quietly mutter under your breath, "Well, that's probably wrong."

[Go to top of section](#)

Multiple Integrals

[Debugging](#)

Multiple integrals show up all the time in physics problems, things like $\int_0^1 \int_0^1 \operatorname{erf}(x \sin(y)) dx dy$.

A double integral like this can be done in Maple with nested **int** commands, like this

> **restart;**

> **s1:=Int(Int(erf(x*sin(y)),x=0..1),y=0..1);**

> **s1:=value(s1);**

$$s1 := \int_0^1 \int_0^1 \operatorname{erf}(x \sin(y)) dx dy$$

s1 :=

$$\int_0^1 \frac{1}{2} \frac{\csc(y) \left(-2 + 2 e^{(-\sin(y))^2} + 2 \sin(y) \sqrt{\pi} \operatorname{erf}(\sin(y)) \right)}{\sqrt{\pi}} dy$$

As usual, if Maple just gives it back to you as a displayed integral it means it can't do it, but if you want a number you can use **evalf**

> **evalf(s1);**

.2440534496

Or, if you just wanted a number in the first place, you can tell Maple to do the integral numerically using this command

> **evalf(Int(Int(erf(x*sin(y)),x=0..1),y=0..1));**

.2440534496

A variation on this command tells Maple to only compute the integral numerically accurate to 6 (or any other number) of significant figures (the ,6 after the integration range specifies the number of digits.)

```
> evalf(Int(Int(erf(x*sin(y)),x=0..1.,6),y=0..1.,6));
```

.244054

You might want to use this if Maple is taking forever to give you the answer--it will go faster if you ask for fewer significant figures.

>

Problem 3.17

Get the numerical answer to this double integral $\int_1^2 \int_1^2 J_0(\sqrt{xy}) dx dy$.

Triple integrals:

It also often happens that you want to do triple integrals, perhaps like

$\int_0^1 \int_0^1 \int_0^1 \text{erf}(x \sin(yz)) dx dy dz$. The straightforward Maple command to do this is

```
> restart;
```

```
> s1:=int(int(int(cos(x)*erf(x*sin(y*z)),x=0..1),y=0..1),z=0..1);
```

$$s1 := \int_0^1 \int_0^1 \int_0^1 \cos(x) \text{erf}(x \sin(yz)) dx dy dz$$

And when Maple gives it back unevaluated, you could use **evalf** again. But when you do you will notice something rather annoying--Maple takes forever to do it. The problem is that for triple integrals there are so many calculations to do that Maple is just too slow. You need a real number cruncher program to get the job done. And for integrals of dimension 4 or higher, even good number crunchers can take a long time. Go ahead and try **evalf**, but be prepared to stop the computation with the stop sign on the toolbar when you can't take it anymore.

> **evalf(s1);**

.1002650803

But if you use Maple's numerical integration command with a limited number of significant figures you get an answer in a reasonable length of time..

> **evalf(Int(Int(Int(cos(x)*erf(x*sin(y*z)),x=0..1.,4),y=0..1.,4),z=0..1.,4));**

.1003

>

Problem 3.18

Use the triple integral method defined above to get the numerical value of the following triple integral in spherical coordinates

$$\int_0^2 \int_0^\pi \int_0^{2\pi} r^6 \sin(\theta)^3 \cos(r \theta \phi)^2 (1 + \sin(\theta)^2) dr d\theta d\phi .$$

Make sure your answer is accurate to 4 significant figures. Even this level of accuracy will take some time to achieve. It would be worth your while to try to find a way to do one of these integrals analytically so that you only have to do a double integral instead of a triple one.

Another way to do double, and even triple, integrals is with the routines in the **student** package. These are a collection of routines written to support a calculus course and you can gain access to them with the command

> **with(student);**

[D, Diff, Doubleint, Int, Limit, Lineint, Product, Sum, Tripleint, changevar, completesquare, distance, equate, integrand, intercept, intparts, leftbox, leftsum, makeproc, middlebox, middlesum, midpoint, powsubs, rightbox, rightsum, showtangent, simpson, slope, summand, trapezoid]

To see what these routines do and how to use them use

> **?student**

[Go to top of section](#)

[Go to top of section](#)

Series expansions

[Debugging](#)

One of the most important mathematical ideas in all of physics is the idea of a series solution to a problem. This is so important in physics because we usually are not interested in the most exact solution to a problem, but rather in the most illuminating solution to a problem. Consider, for example, the function $\operatorname{sech}(x)$. This is a function you should know, and even learned once, but have since forgotten. And even if you happen to remember that it is the reciprocal of the hyperbolic cosine function, this probably doesn't help you to visualize it very well. But what if I were to tell you that near zero this

function is approximated by the simple function $1 - \frac{x^2}{2}$? Now you can see that it is 1 at $x = 0$

and that it falls off like a parabola in both directions from $x = 0$. And then what if I told you that for

$|x|$ very large the function is approximated by the function $2e^{-|x|}$? Now you can see that it

falls to zero exponentially away from $x = 0$, so the function just looks like a haystack.

Maple knows how to find approximations to most of the functions it deals with. The commands that find

these approximations are **taylor** , **series** , and **asympt** ; we will see how they work in this section.

Taylor's theorem is: that if a function $f(x)$ is defined and has well-defined derivatives at the point $x = a$, then for x near a the function can be approximated by the series

$$f(x) = f(a) + \left(\frac{\partial}{\partial x} f(a) \right) (x - a) + \frac{\left(\frac{\partial^2}{\partial x^2} f(a) \right) (x - a)^2}{2!} + \frac{\left(\frac{\partial^3}{\partial x^3} f(a) \right) (x - a)^3}{3!} + \dots$$

Maple knows this theorem and can use it to generate all kinds of series with the command **taylor** . For instance, here is the one for the **sech** function discussed above

> **taylor(sech(x),x=0,20);**

$$1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \frac{61}{720}x^6 + \frac{277}{8064}x^8 - \frac{50521}{3628800}x^{10} + \frac{540553}{95800320}x^{12} - \frac{199360981}{87178291200}x^{14} + \frac{3878302429}{4184557977600}x^{16} - \frac{2404879675441}{6402373705728000}x^{18} + O(x^{20})$$

The $x=0$ option tells Maple to take the expansion about 0 and the 20 tells it to include terms until the error term is of order x^{20} .

>

Problem 3.19

Find the Taylor expansions of the following functions about $x = 0$ and keeping terms out to x^{10} .

(a) $\sin(x)$ (b) $\cos(x)$ (c) $\arctan(x)$ (d) e^x (e) $\ln(1+x)$ (f) $(1+x)^p$

Your answers will look pretty impressive, but you will find that in most physics problems where expansions are important, we only keep the first two terms, so you can often get what you want without all of the clutter by using **taylor(f(x),x=0,3)**, which just goes through x^2 .

Problem 3.20

Maple can also do Taylor expansions of special functions. Find the expansions of (a) $J_0(x)$, (b) $I_2(x)$, (c) $K(x)$ (complete elliptic integral **EllipticK**), and (d) $\Gamma(x)$. Do (a)-(c) about $x=0$ and (d) about $x=1$. Do each one through x^{10} . The answer to (d) is pretty awful--see if you can get something more reasonable by using **evalf**, or by telling Maple to expand about $x=1.0$ (decimal point again).

Problem 3.21

Now try expanding the Bessel function $K_0(x)$ about $x=0$. Maple will fail, but it will make a suggestion, and it is a good one. If the function in question has a singularity of some kind at the expansion point, then either the function or some of its derivatives do not exist at that point and a Taylor series is not possible. But there are other kinds of series that can approximate functions, and Maple's **series** command, which works almost like **taylor** (but is more general) can give them to you. Try **series(BesselK(0,x),x=0,3)**; and see what happens. Notice that by ignoring most of this mess and concentrating just on the leading term you can tell that $K_0(x)$ is logarithmically singular. (Notice also that Euler's constant γ appears in this expansion. To learn a little about it use **?gamma**.) When you get this to work, try the same thing on the following functions:

$$(a) \cot(x) \quad (b) \frac{1}{\tanh(x)} \quad (c) \frac{\sqrt{x}}{\sin(x^2)} .$$

Try different numbers in place of the 3 in the series command above and see if you can tell precisely what it means.

Problem 3.22

There is another kind of expansion that is very useful in mathematical physics: the asymptotic expansion. It is an expansion about ∞ , which seems odd, but all it means is that we want to know how a function behaves when its argument is really big. The Maple command that does this is **asympt(f, x, 2)** (or **series(f, x=infinity, 2)**) where f is a Maple expression. Try this on the following functions using **series** on (a)-(b) and **asympt** on (c)-(d). Only keep 2 terms, as indicated in the sample commands just above.

$$(a) J_0(x) \quad (b) I_1(x) \quad (c) K_0(x) \quad (d) \Gamma(x) .$$

I have suggested that you only put a 2 in the last slot because usually all we care about in an asymptotic expansion is the leading term because it answers the most important questions: (1) does the function blow up or die? and (2) in what way does it do it?

Problem 3.23

Here's an example of how we use this idea to solve physics problems. The equation of motion of a pendulum is

$$\frac{\partial^2}{\partial t^2} \theta = - \frac{g \sin(\theta)}{L}$$

where the angle θ is zero if the pendulum hangs straight down and is π if the pendulum is balancing straight up (think a rod pendulum, not a string one).

Maple can solve this equation, but the answer looks both ugly and unfamiliar. So what we would like to do is look at this equation near some special values of θ (like straight down: $\theta = 0$ and straight up: $\theta = \pi$) and see if we can change this hard differential equation into an approximate easier one. (Note the rest of this problem will only make sense if you know what the solutions of these two differential equations look like:

$$(i) \frac{\partial^2}{\partial t^2} x = -\omega^2 x \quad \text{and} \quad (ii) \frac{\partial^2}{\partial t^2} x = \omega^2 x .$$

The first one is the simple harmonic oscillator whose solutions just wiggle up and down sinusoidally at frequency ω like this: $\cos(\omega t)$ or $\sin(\omega t)$; the second one has a growing exponential $e^{(\omega t)}$ for one of its solutions, so it represents unstable motion. Remember these two forms, one

with a positive coefficient on the right and one with a negative one as you continue with this problem.)

The pendulum equation with $\sin(\theta)$ in it can be made to look like these simpler equations by using a series expansion. (a) Expand the right-hand side of the pendulum equation about the downward equilibrium position at $\theta = 0$ and by comparing it to (i) and (ii) show that simple harmonic oscillation results. (b) Expand the right-hand side of the pendulum equation about the vertical equilibrium position at $\theta = \pi$. After you expand make the u-substitution $\theta = \phi + \pi$ in the pendulum equation

$$\frac{\partial^2}{\partial t^2} \theta = - \frac{g \sin(\theta)}{L}$$

to get a differential equation in ϕ . Compare the resulting equation with (i) and (ii) to show that this new equation has unstable solutions. (This is really just a fancy mathematical way of saying something you already know: if you hang a yardstick straight down and wiggle it a little, it just swings back and forth; if you balance a yardstick straight up and wiggle it a little it falls over.)

Problem 3.24

In Einstein's theory of relativity the energy of an object of mass m moving at speed v is given by

the famous formula $E = \frac{m c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$. An interesting question is: what happened to good old kinetic energy $K = \frac{m v^2}{2}$ from freshman physics? The answer is obtained by means of a Taylor

expansion. Assume that the object is moving at much less than the speed of light and make a series expansion of Einstein's formula in v , keeping two terms. One of them will be your old friend and the other is Einstein's idea of rest energy: objects with mass have energy.

Problem 3.25

The formula from freshman electricity and magnetism for the potential (voltage) of a point charge is

$V(r) = \frac{q}{4 \pi \epsilon_0 r}$. The formula for the potential of a dipole is usually not introduced until later, but

Maple can help you find it. A dipole is a pair of point charges, one with charge q and the other with charge $-q$, separated by distance d . Their dipole moment is denoted by the symbol P with magnitude $p = qd$. Let's place the positive charge on the z-axis at $\frac{d}{2}$ and the negative charge on

the z-axis at $-\frac{d}{2}$. If we do this and use spherical coordinates then the potential as a function of distance r from the origin and polar angle θ is

$$V(r, \theta) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{r^2 + \frac{d^2}{4} - rd\cos(\theta)}} - \frac{1}{\sqrt{r^2 + \frac{d^2}{4} + rd\cos(\theta)}} \right)$$

Make a series expansion of this formula in small separation distance d and just keep the leading term to find the formula for the potential of a dipole in spherical coordinates. Help Maple out by using **assume** ($r>0$). You may have to experiment with the order of the expansion--1 won't get the job done. You may also find it helpful to use the 'leadterm' form of the series command. Look it up in online help using **? series[leadterm]**.

[Go to top of section](#)

Sums

[Debugging](#)

Your calculus class was also where you got introduced to the idea of infinite sums. Maple knows all about them and can do some pretty amazing things. Consider my personal favorite $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The

Maple command that does this is **sum**

and its syntax is about the same as **int's** syntax: (Note: **Sum** is the inert form.) Try

> **restart;**

```
> sum(1/'n'^2,'n'=1..infinity);
```

$$\frac{1}{6}\pi^2$$

a lovely and unexpected result. In this form the first argument of **sum** is an expression. Note a peculiarity of Maple in this command: the index n is in single quotes. You don't always have to do this, but if you don't then Maple will fail you at some point in your computational life, so you ought to at least know about this use of single quotes. (The single quotes tell Maple to use this as a dummy variable in this command ignoring any possible previous assignment of something to the variable n .) Having made you aware of this feature, I'm going to ignore it because typing quotes is a pain.

The **sum** command, and its inert form **Sum**, also work with functions.

```
> fn:=n->1/n^2;
```

$$fn := n \rightarrow \frac{1}{n^2}$$

```
> s1:=Sum(fn(n),n=1..infinity);
```

$$s1 := \sum_{n=1}^{\infty} \frac{1}{n^2}$$

```
> value(s1);
```

$$\frac{1}{6}\pi^2$$

```
>
```

To see what the possibilities are for doing sums in Maple are, take a minute and look at **?sum**. Here are some exercises for practice.

Problem 3.26

Find each of these sums using expression form for (a)-(c) and function form for (d)-(f).

$$(a) \sum_{n=1}^N n^3 \quad (b) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad (c) \sum_{n=1}^{\infty} \frac{2n+3}{(n+1)(n+2)(n+3)}$$

$$(d) \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N \frac{1}{n} \right) - \ln(N) \quad . \text{ Note: } N \text{ goes to infinity in both terms, not just in the sum. The}$$

funny little symbol you will get is not a mistake, but a very famous number denoted by the symbol γ (gamma). It is called Euler's constant. Check how big it is with **evalf**.

$$(e) \sum_{n=1}^{\infty} \frac{x^n}{n} \quad (f) \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$$

[Go to top of section](#)

[Go to top of chapter](#)

>

[Chapter4.mws](#)

- [Complex arithmetic](#)
- [Elementary complex functions](#)
- [Special complex functions](#)
- [Wave interference](#)
- [Electrostatics with line charges](#)

Chapter 4: Complex numbers and functions

You first learned about complex numbers when an algebra teacher tried to convince you that $\sqrt{-1}$ made sense. It doesn't just make sense; it is a powerful way to model the real world we live in, in spite of i being a so-called "imaginary number". Maple knows all about the complex plane, starting with a special variable to represent $\sqrt{-1}$: Maple calls it I ; yes, capital **I**. So in Maple the complex number $5 + 3i$ would be written this way

```
> z1:=5 + 3*I;
```

```
z1 := 5 + 3 I
```

```
>
```

You need to be careful with this syntax because Maple will allow you to use i , like this:

```
> z2:=5+3*i;
```

```
z2 := 5 + 3 i
```

This looks fine, but when you do arithmetic with " i " it won't work right because i is just another variable that doesn't have a value yet. Only I is the real thing. (Actually, you can redefine I to be i if you want by using the **interface** command, but I suggest you don't.)

```
>
```

In this chapter we will review the various things you can do in the complex plane, including arithmetic,

calculus, and even special functions like $J_0(z)$ and $\Gamma(z)$. Note: in this chapter whenever you encounter a variable with a name like z , or $z1$, or $z2$, etc., it is a complex variable of the form $z = x + Iy$.

Complex arithmetic

[Debugging](#)

Let's review what you can do with complex numbers. First, there is simple addition and subtraction. And since the rule is that real parts add and imaginary parts add, if you think about the real numbers being the x -axis and the imaginary numbers being the y -axis, together forming the complex xy -plane, then it is not too hard to see that complex addition and subtraction are just addition and subtraction of vectors in this 2-dimensional plane. And Maple knows how to do it:

```
> z1:=5+3*I;
```

$$z1 := 5 + 3 I$$

```
> z2:=1-6*I;
```

$$z2 := 1 - 6 I$$

```
> z3:=z2+z1;
```

$$z3 := 6 - 3 I$$

Just for fun, here's how this addition problem looks using arrows in the complex plane. (Note: I am showing you some of Maple's tools for drawing pictures in the commands below, so there is more useful information here than just the discussion of complex arithmetic). In the pictures $z1$ is green, $z2$ is blue, and $z3$ is red.

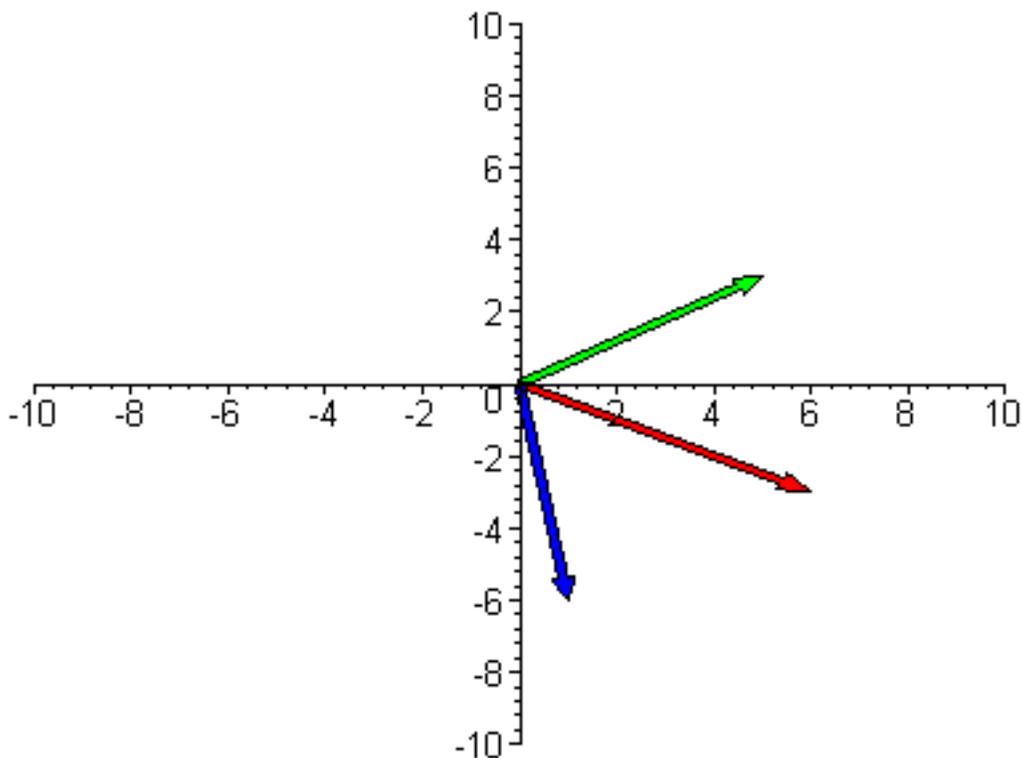
```
> with(plottools):
```

```
l1 := arrow([0,0], [5,3], .2, .5, .1, color=green):
```

```
l2 := arrow([0,0], [1,-6], .2, .5, .1, color=blue):
```

```
l3 := arrow([0,0], [6,-3],.2,.5,.1,color=red):
```

```
> plots[display](l1,l2,l3, axes=normal,view=[-10..10,-10..10]);
```



Ok: addition and subtraction are easy. Multiplication and division, however, are not anything like the vector products you know about from your physics classes. The complex multiplication rule is simple enough: just treat the real and imaginary parts like variables in algebra and multiply everything out,

remembering that $i^2 = -1$. Here, for example is a multiplication:

$(5+3i)*(1-6i) = (5 + 3i - 30i + 18) = 23-27i$. You just use the distributive law and work it out. And Maple knows how to do it:

> **z1*z2;**

$23 - 27i$

Division is a little trickier because i appears in the denominator. We take care of this problem with the idea of the complex conjugate. The complex conjugate of z_1 is almost the same as z_1 , except that the sign of the imaginary part of z_1 has its sign changed, like this:

> **conjugate(6+4*i);**

$6 - 4i$

The conjugate is useful because of what the product of a complex number and its conjugate turn out to

be:

> **conjugate(6+4*I)*(6+4*I);**

52

> **assume(x,real);assume(y,real);**

> **conjugate(x+y*I)*(x+y*I);**

$$(x - Iy)(x + Iy)$$

> **expand(%);**

$$x^2 + y^2$$

Their product is just a real number, equal to the square of the magnitude of the complex number, treated as a vector in the usual Pythagorean way: square the components and add. The answer is

$6^2 + 4^2 = 52$. This is useful for doing division because if you multiply the top and bottom of a

complex fraction by the conjugate of the bottom, the bottom becomes real and you don't have to worry about I in the denominator anymore. Just multiply the numerator out using the distributive law and divide both real and imaginary parts by the real number in the bottom.

>

Problem 4.1

Work out by hand the complex division problem $\frac{3 + 4I}{2 - 5I}$, then do it with Maple and show that the

answers agree.

Ok, this is simple enough to do, but what does it mean? Is there a picture in the complex plane that allows us to visualize what multiplication and division mean? Well, there is; but to understand it we need to have another way of thinking about complex numbers. When we write $5+4I$ we are using

Cartesian coordinates in the complex plane. But it is useful to use polar coordinates in the complex plane as well. In polar coordinates the number $5+4i$ would be represented by a radius of magnitude

$\sqrt{5^2 + 4^2} = \sqrt{41}$, and an angle θ measured counterclockwise from the real axis, with

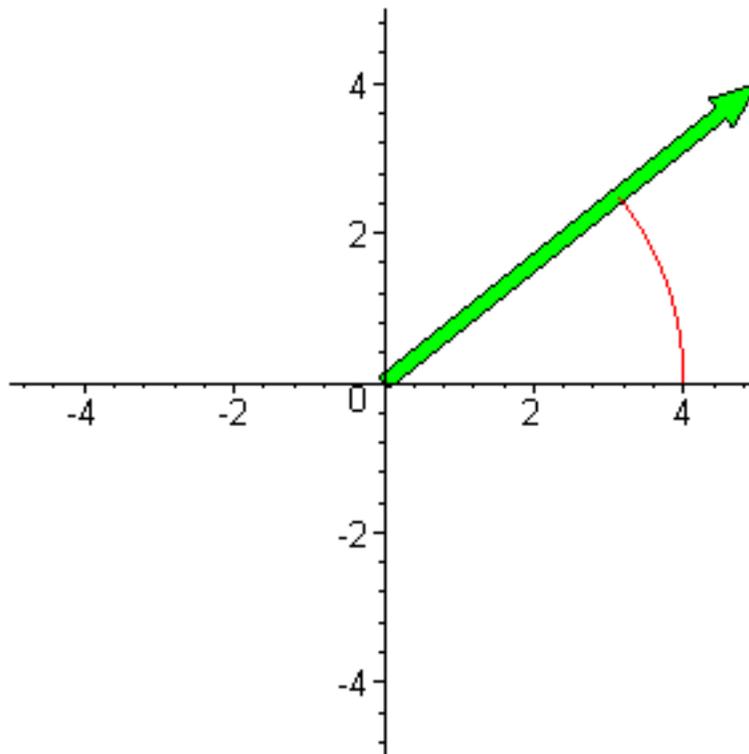
$$\theta = \arctan\left(\frac{4}{5}\right), \text{ like this}$$

> **with(plottools):**

l1 := arrow([0,0], [5,4], .2, .5, .1, color=green):

l2 := arc([0,0],4,0..arctan(.8),color=red):

> **plots[display](l1,l2, axes=normal,view=[-5..5,-5..5],scaling=constrained);**



>

Polar coordinates are a wonderful alternate way to visualize complex numbers. The connection between the complex number represented by $x + iy$ and the polar form (r, θ) is just a simple exercise in trigonometry: $x = r \cos \theta$ and $y = r \sin(\theta)$. The Swiss mathematician Leonhard Euler discovered that the exponential function applied to a purely imaginary number gives the remarkable

result $e^{i\theta} = \cos(\theta) + i \sin(\theta)$. This means that the connection between the Cartesian and polar

forms of complex numbers can be written in this compact form: $x + iy = r e^{i\theta}$. And this form,

combined with the properties of the exponential function under multiplication and division, allows us to interpret what complex multiplication and division mean. Let's look at multiplication first. Writing

$z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ allows us to see that their product can be written as

$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$. So this is what multiplication means: the product has a magnitude

which is the product of the magnitudes of z_1 and z_2 and an angle which is the sum of the angles of

z_1 and z_2 , as shown in the picture below. The two complex numbers z_1 and z_2 are in green and their product is in red.

> **with(plottools):**

l1 := arrow([0,0], [1,2], .1, .3, .1, color=green):

l1a := arc([0,0],1.5,0..arctan(2),color=green):

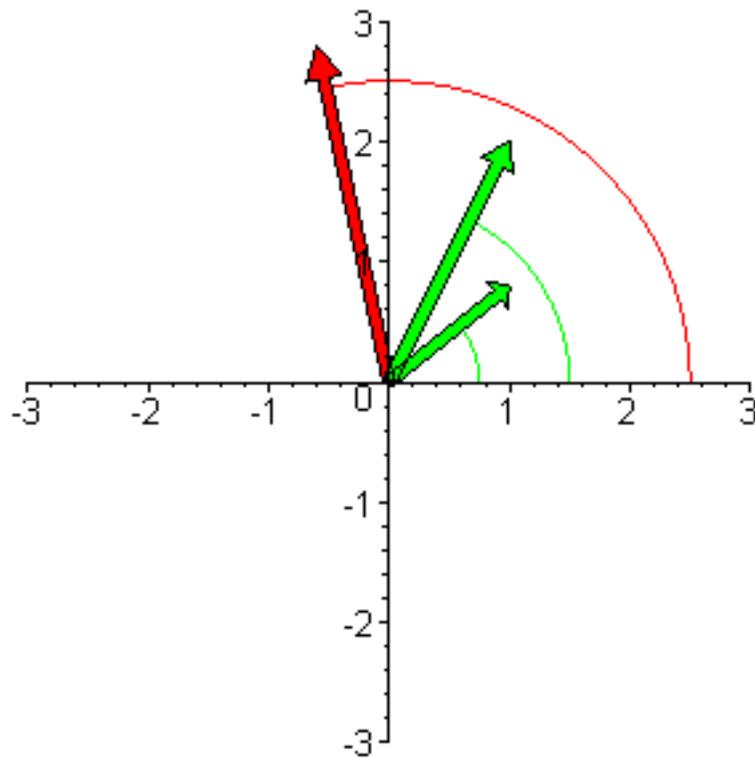
> **l2 := arrow([0,0], [1,.8], .1, .3, .1, color=green):**

l2a := arc([0,0],.75,0..arctan(.8),color=green):

> **l3 := arrow([0,0], [-.6,2.8], .1, .3, .1, color=red):**

l3a := arc([0,0],2.5,0..arctan(2.8,-.6),color=red):

> **plots[display](l1,l2,l3,l1a,l2a,l3a, axes=normal,view=[-3..3,-3..3],scaling=constrained);**



>

Division works similarly, but with the two magnitudes being divided and the two angles subtracting:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i(\theta_1 - \theta_2)}}{r_2}$$

Well, that's about it for the simple arithmetic. But before proceeding onto more interesting stuff I should mention that Maple has some commands that come in handy when working with complex numbers. **Re(z)** picks off the real part of z, **Im(z)** gives you the imaginary part, **abs(z)** gives you the polar magnitude, and **arctan(Im(z),Re(z))** gives you the angle, like this

> **z:=5-3*I;**

$$z := 5 - 3I$$

> **Re(z);Im(z);abs(z);arctan(Im(z),Re(z));**

$$-3$$

$$\sqrt{34}$$

$$-\arctan\left(\frac{3}{5}\right)$$

Maple also has a command that converts z in Cartesian form into polar form, but I find it hard to use its results in subsequent calculations, so I usually just use the arctan form given above. But here it is anyway.

> **readlib(polar):**

> **polar(z);**

$$\text{polar}\left(\sqrt{34}, -\arctan\left(\frac{3}{5}\right)\right)$$

And finally, remember that Maple will take the complex **conjugate** for you.

> **conjugate(3+4*I);**

$$3 - 4 I$$

>

Problem 4.2

Do the operations indicated below on the complex numbers z_1 and z_2 and express your answers in both Cartesian ($x+iy$) form and polar form (**readlib(polar):** and the **polar** command make this problem easy).

Here are the definitions of z_1 and z_2 that you are supposed to use in this problem:

> **z1:=3-5*I;z2:=-2+3*I;**

$$z1 := 3 - 5 I$$

$$z2 := -2 + 3 I$$

(a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $z_1 z_2$ (d) $\frac{z_1}{z_2}$

>

Problem 4.3

Show graphically, by making a Maple arrow plot like the one shown above, that rotating a complex number through angle θ in the complex plane is easily accomplished by multiplying by the factor

$e^{(i \theta)}$. Do this by defining a complex number $z1$ and plotting it as an arrow, then multiplying $z1$ by

$e^{(I \theta)}$ for some angle θ that you choose, and then by plotting this new complex number as an

arrow. As you do this you need to be careful to only use floating point numbers in the arguments of the **arrow** command. Maple will probably give you answers back that have square roots of integers in them and if you just copy these numbers into the arrow command no arrow will appear. Our old friend **evalf** will help you here, as will this syntax for putting the real and imaginary parts of a complex number into the arrow command: **[Re(z),Im(z)]**.

[Go to top of section](#)

Elementary complex functions

[Debugging](#)

All of the functions you have ever heard of make sense for complex numbers as well as for real numbers. And in fact, the complex plane can give a more unified picture of these functions than is possible when we are confined only to the real axis.

We will begin with the exponential function. Euler's formula $e^{(i\theta)} = \cos(\theta) + i \sin(\theta)$ and the

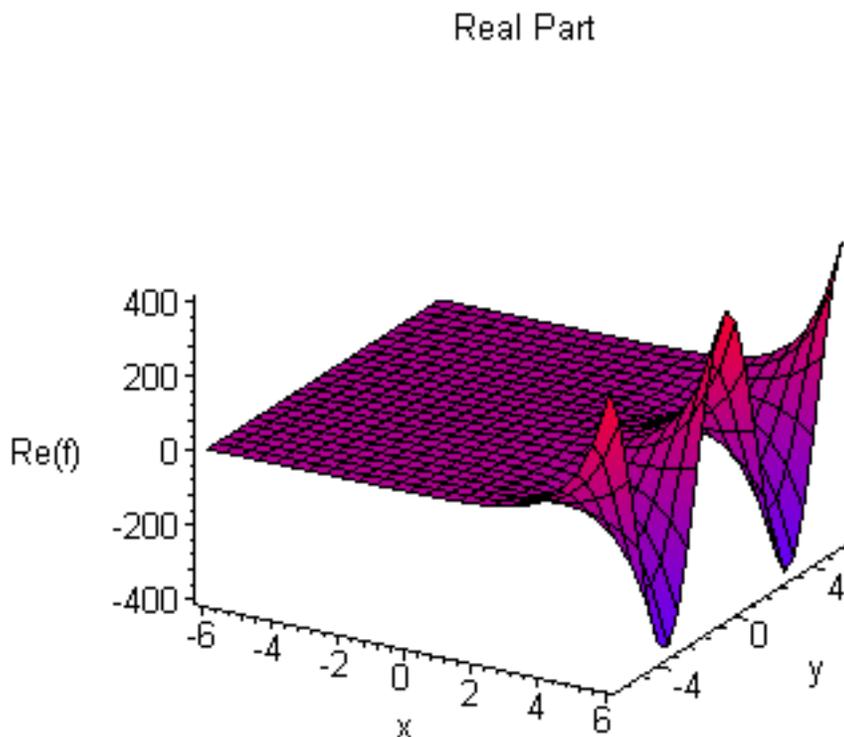
properties of the exponential function allow us to write for e^z (with $z = x + iy$),

$e^{(x+iy)} = e^x (\cos(y) + i \sin(y))$. You may not be able to visualize what this function looks

like just by staring at this formula, so to see the behavior of this function over the entire complex plane, let's make a 3-D surface plot of its real part and its imaginary part using Maple's **plot3d** command (see [3d plot](#)).

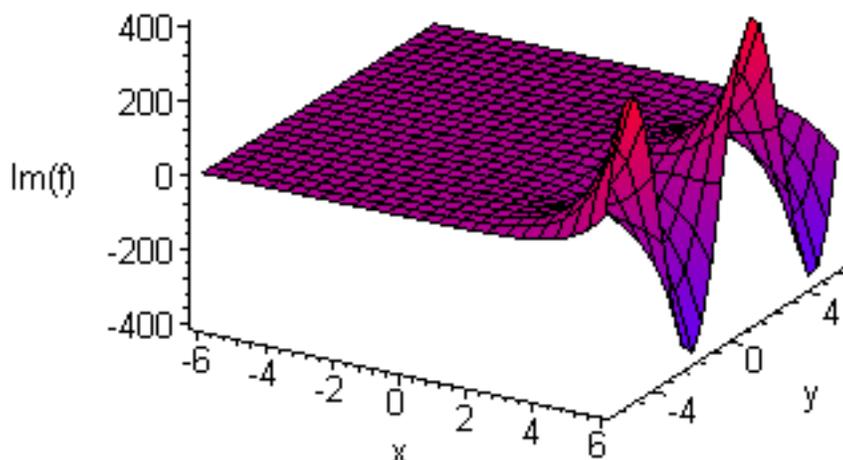
> **restart;**

> **plot3d(Re(exp(x+I*y)),x=-6..6,y=-6..6,shading=z,axes=framed,labels=["x","y","Re(f)"],orientation=[-60,60],title="Real Part");**



> **plot3d(Im(exp(x+I*y)),x=-6..6,y=-6..6,shading=z,axes=framed,labels=["x","y","Im(f)"],orientation=[-60,60],title="Imaginary Part");**

Imaginary Part



Notice that at negative x it is very small and for positive x it is very big--this is the behavior of e^x . In y , the functions just oscillate like cosines and sines, which is, of course, exactly what they are.

This mixing together of the exponential function and sines and cosines gives rise to several interesting formulas which you are to verify in the next problem.

>

Problem 4.4

Use Maple to verify the following functional identities. Do this problem by going to **?evalc**, reading it over and looking at the examples, and then using this command to check these identities.

$$(a) \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad (b) \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad (c) \cos(ix) = \cosh(x) \quad (d)$$

$$\sin(ix) = i \sinh(x)$$

>

Problem 4.5

Make 3-D surface plots over the complex plane of the real parts of $\cos(z)$ and $\sin(z)$, and of the imaginary part of $\tan(z)$, where $z = x + iy$. Use x in the range $-9..9$ and y in the range $-5..5$ for \sin and \cos ; use $-5..5$ and $-3..3$ for \tan . Stare at them until they make sense. When you do the tangent function you will have to deal with its singularities. Do this by using the view option in the plot, in the form `view=-5..5`. This vertical size control keeps Maple from trying to plot infinity. It may also help to have a finer grid, which you can get with `grid=[30,30]`. To make sense of it you will also have to remember what the \tanh function looks like.

>

Now let's move on to power-law functions, i.e., we want to study the behavior of z^p in the complex plane. It is easier to see what this function does if we use the polar representation: $z^p = r^p e^{(ip\theta)}$.

This behavior is not too hard to visualize for integer powers p . The magnitude is just raised to the p power and the imaginary exponential just wiggles faster as we go around in polar angle.

Problem 4.6

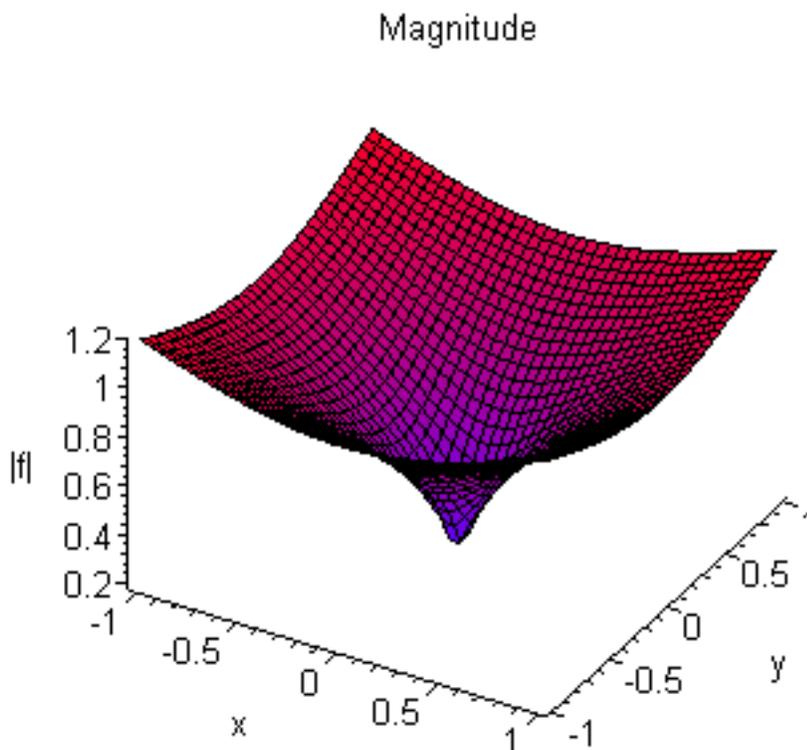
Make a 3-D plot of the real parts of the functions z^2 and z^5 using the range $-3..3$ in both x and y for z^2 and a smaller range for z^5 .

For non-integer powers things get a lot more complicated. To see why, let's consider the square root

function, $f = \frac{1}{2}$. Below I have coded three 3-D plots of this function. The first is of its magnitude,

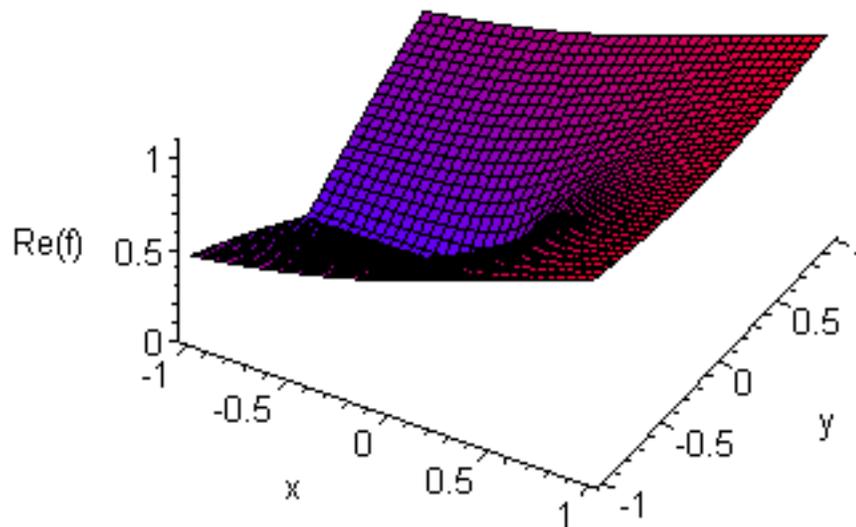
and it just looks like the normal square root function rotated about the z-axis, so this looks like we expected. But there is something funny going on along the negative x-axis in the real part plot, and it gets worse in the imaginary part plot.

```
> plot3d(abs((x+I*y)^(1/2)),x=-1..1,y=-1..1,shading=z,axes=framed,labels=["x","y","|f|"],
orientation=[-60,45],title="Magnitude",grid=[40,40]);
```



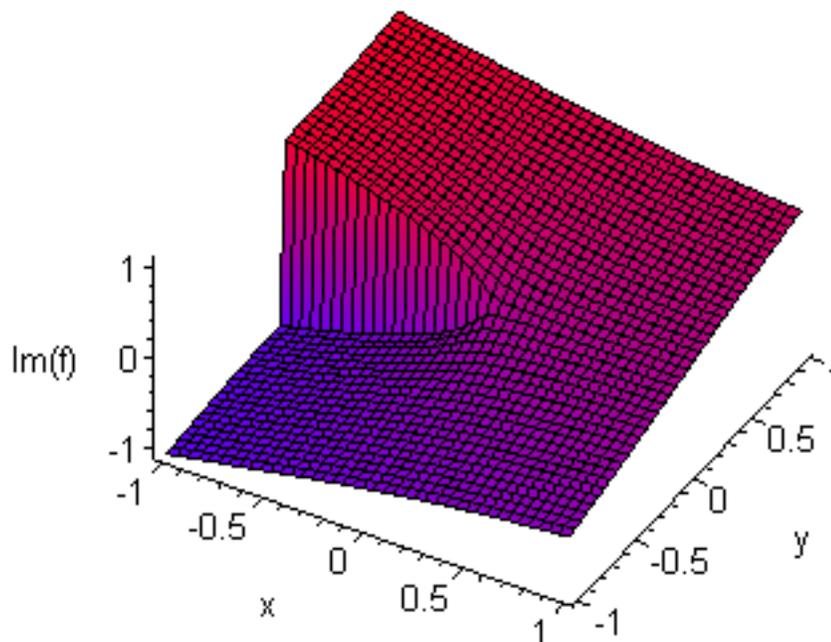
```
> plot3d(Re((x+I*y)^(1/2)),x=-1..1,y=-1..1,shading=z,axes=framed,labels=["x","y","Re(f)"],
orientation=[-60,35],title="Real Part",grid=[40,40]);
```

Real Part



```
> plot3d(Im((x+I*y)^(1/2)),x=-1..1,y=-1..1,shading=z,axes=framed,labels=["x","y","Im(f)"],
orientation=[-60,35],title="Imaginary Part",grid=[40,40]);
```

Imaginary Part



Surely ordinary functions can't behave this way!?! Well, they can in the complex plane. Here's the problem. Maple defines its polar angle θ to be in the range $-\pi .. \pi$. Let's see what the square root

function does to these two different values of z : $z_1 = e^{(i \pi)}$ and $z_2 = e^{(-i \pi)}$. The first thing to

notice is that z_1 and z_2 are exactly the same number, namely -1. The second thing to notice is that the square root function turns the first one into i and the second one into $-i$! So the square root of -1 has different answers depending on what angle we use to get to -1 in the complex plane. This is the origin of the cliff face in the imaginary part plot, and the line in the complex plane where the cliff occurs is called a branch cut. Branch cuts are a pain, and you will study them in more detail when you take a course in complex analysis. All fractional powers have this problem, as does the plain old ordinary log function, $\ln(z)$, as you will soon discover.

>

Problem 4.7

Do the same thing to the function $\ln(z)$ that we just did to square root. Locate the branch cut and explain why it's there by looking at the same problem discussed in the paragraph above. Note: be sure to use the function $\ln(x + Iy)$ in your plot commands.

Problem 4.8

Finally, do a set of 3-D surface plots of this rather ordinary looking and perfectly well-behaved-along-

the-real-axis function $f(z) = \frac{z^2}{1+z^2}$ (remember that $z = x + Iy$). First make a plot of $f(x)$ along

the real axis so you know what it looks like there. Use the x-range -4..4. Then make surface plots of the real part, the imaginary part, and the magnitude (use **abs**), again using -4..4 in both dimensions. Also use the option `view=-3..3`. Hopefully this will give you a sense of the surprise and beauty of the complex plane.

[Go to top of section](#)

Special complex functions

[Debugging](#)

It will probably come as no surprise that the more advanced functions of mathematical physics do the same kinds of things in the complex plane that the elementary ones do. Rather than talking about it I'll let you discover a few of the interesting things that happen through these exercises.

Problem 4.9

Use Maple to show that the Bessel functions $J_0(z)$ and $I_0(z)$ are related in the complex plane by asking Maple to evaluate $J_0(ix)$ and $I_0(ix)$. Also do it for a few more orders of Bessel functions, say $n=1,2,3$. Then try the same thing for the Bessel function Y_0 . In this second case Maple will not tell you anything interesting, even though there is something interesting to be told. It would be nice if it would tell you that $Y_0(ix) = i I_0(x) - \frac{2 K_0(x)}{\pi}$, but I don't know how to get it to do it.

Just as a check, at least, try verifying this identity numerically to see if Maple knows what it's doing. Use $z = ix$ for a few real values of x .

Problem 4.10

Do a 3-D plotting exploration of the behavior of the elliptic integral $K(z)$ in the complex plane. This function is obtained with the Maple command **EllipticK(z)**. Look at its real part and imaginary part with both x and y in the range $-2..2$.

[Go to top of section](#)

Wave interference

[Debugging](#)

There are many ways that the complex plane helps us to do physics; perhaps the most important is in the study of phase shifts and interference patterns. This is probably a good time to give you a simple definition of the technical term *phase*. Phase is anything that appears as the argument of a sine or cosine function. For instance, if an oscillation in time is described by the function $\cos(\omega t)$, then we say that the phase of the oscillation is changing linearly in time and is simply given by ωt . If the

oscillation is shifted in time so that it is given instead by $\cos(\omega t + \phi)$, then the new phase of the oscillation is $\omega t + \phi$, and we say that relative to the previous oscillation, this one has a phase shift ϕ . Phase shifts can be a real pain to handle because they are stuck inside cosines and sines and you have to know lots of trig identities to get simple answers out. But if we remember that

$$\cos(\omega t) = \Re(e^{(i \omega t)}) \quad (\text{that funny script R is Maple's pretty representation of } \mathbf{Re}, \text{ the real part})$$

then lots of formulas can be simplified. Here's an example. Suppose we have 12 different oscillations going like $\cos(\omega t + \phi_n)$, all interfering with each other by adding together and suppose also that

the n th wave has a phase shift given by $\phi_n = \frac{2 n \pi}{13}$. Using the power of Maple we can add these

oscillations together to get

> **restart;**

> **s1:=Sum(cos(omega*t+2*Pi*n/13),n=1..12);**

$$s1 := \sum_{n=1}^{12} \cos\left(\omega t + \frac{2}{13} \pi n\right)$$

> **s1:=value(s1);**

$$\begin{aligned} s1 := & \cos\left(\omega t + \frac{2}{13} \pi\right) + \cos\left(\omega t + \frac{4}{13} \pi\right) + \cos\left(\omega t + \frac{6}{13} \pi\right) \\ & - \sin\left(\omega t + \frac{3}{26} \pi\right) - \sin\left(\omega t + \frac{7}{26} \pi\right) - \sin\left(\omega t + \frac{11}{26} \pi\right) \\ & - \cos\left(\omega t + \frac{1}{13} \pi\right) - \cos\left(\omega t + \frac{3}{13} \pi\right) - \cos\left(\omega t + \frac{5}{13} \pi\right) \\ & + \sin\left(\omega t + \frac{1}{26} \pi\right) + \sin\left(\omega t + \frac{5}{26} \pi\right) + \sin\left(\omega t + \frac{9}{26} \pi\right) \end{aligned}$$

> **simplify(s1);**

$$\begin{aligned} & \cos\left(\omega t + \frac{2}{13}\pi\right) + \cos\left(\omega t + \frac{4}{13}\pi\right) + \cos\left(\omega t + \frac{6}{13}\pi\right) \\ & - \sin\left(\omega t + \frac{3}{26}\pi\right) - \sin\left(\omega t + \frac{7}{26}\pi\right) - \sin\left(\omega t + \frac{11}{26}\pi\right) \\ & - \cos\left(\omega t + \frac{1}{13}\pi\right) - \cos\left(\omega t + \frac{3}{13}\pi\right) - \cos\left(\omega t + \frac{5}{13}\pi\right) \\ & + \sin\left(\omega t + \frac{1}{26}\pi\right) + \sin\left(\omega t + \frac{5}{26}\pi\right) + \sin\left(\omega t + \frac{9}{26}\pi\right) \end{aligned}$$

Well, that's an answer, but it's not very pretty, and it's tough to get Maple to simplify it.

>

Now let's try doing it using the complex plane. Instead of adding 12 cosines together we'll add 12 complex exponentials together. Since the real part of each one is a cosine we can do the complex sum, then take the real part at the end and it will be as if we had added 12 cosines together. The only difference is that the answer will be a lot prettier. Ok, let's see how this goes. Rather than giving the problem to Maple right away, let's do a little pre-analysis first. Here's the sum whose real part we will take at the end

$$\sum_{n=1}^{12} e^{(i\omega t + i\phi_n)}$$

We will take out the common factor of $e^{(i\omega t)}$ (*notice: time factored out! this is the whole reason*

for using the complex exponential) and then rewrite the remaining complex exponential this way

$$e^{\left(\frac{i 2 \pi n}{13}\right)} = \left(e^{\left(\frac{i 2 \pi}{13}\right)}\right)^n$$

. So now our sum looks like this

$$e^{(i \omega t)} \left(\sum_{n=1}^{12} z^n \right) \text{ where } z = e^{\left(\frac{i 2 \pi}{13}\right)}$$

. Now Maple can finish it off.

> **z:=exp(I*2*Pi/13);**

$$z := e^{\left(\frac{2}{13} I \pi\right)}$$

> **s:=Sum(z^n,n=1..12);**

$$s := \sum_{n=1}^{12} \left(e^{\left(\frac{2}{13} I \pi\right)} \right)^n$$

> **s:=simplify(value(s));**

$$s := -1$$

Finally, we tack on the $e^{(i \omega t)}$ we factored out front at the beginning, take the real part, and

discover that the big mess of cosine functions in the original Maple sum is just:

$$\sum_{n=1}^{12} \cos\left(\omega t + \frac{2\pi n}{13}\right) = -\cos(\omega t)$$

The lesson here is hopefully clear; if you are doing an interference problem where you have to add up a whole bunch of sines or cosines, do it with complex exponentials instead. Also, don't be discouraged if Maple doesn't give you a simple answer right away. Use what you know about complex numbers to give Maple some extra help.

>

Problem 4.11

(Note: this isn't really a problem. Just follow along, execute all of the Maple commands, and pay close attention; some real problems will follow.) Let's do a simple interference problem using complex analysis. The two-slit diffraction pattern is always taught in Physics 221, but only in the far field approximation. We'll use Maple to do it right. We'll have two sources of light a distance d apart and perpendicular to the line between them at distance L is a screen. We want to find the intensity of a wave on the screen as a function of distance y on the screen, measured upward from the midline from between the sources to the screen, as shown in the plot below. (Note that this shows you how to draw figures in Maple. Use **?plottools** to find more drawing stuff.)

- > **restart:with(plots):**
- > **l1:= listplot([[0,0],[2,0]]): # distance to screen**
- > **t1:= textplot([1,-.1,"L"]): # L label**
- > **l2 := listplot([[-.08,-.2],[-.1,-.2],[-.1,.2],[-.08,.2]]): # d bracket**
- > **t2:= textplot([-0.15,0,"d"]): # d label**
- > **p1:= pointplot([0,-.2],symbol=circle): # lower source**
- > **p2:= pointplot([0,.2],symbol=circle): # upper source**
- > **l3:= listplot([[2,-1.25],[2,1.25]]): # screen**

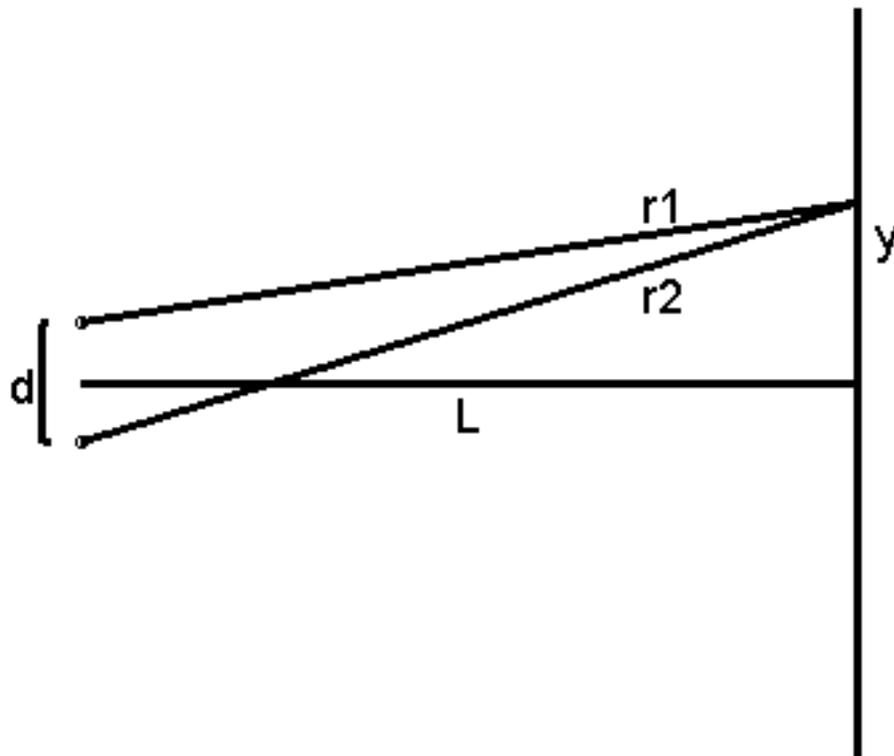
```

> t3:= textplot([2.08,.5,'y']): # y label
> l4:= listplot([[0,.2],[2,.6]]): # upper ray
> t4:= textplot([1.5,.6,'r1']): # r1 label
> l5:= listplot([[0,-.2],[2,.6]]): # lower ray
> t5:= textplot([1.5,3,'r2']): # r2 label

> plots[display]([l1,t1,l2,t2,p1,p2,l3,t3,l4,t4,l5,t5],axes=none,thickness=3,view=[-.5..2.1,-1.25..1.25],
font=[HELVETICA,14]);

```

Warning, the name changecoords has been redefined



To study the interference of these two sources with as much accuracy as possible we will let the field

from each one fall off like $\frac{e^{i(kr - \omega t)}}{r}$ from the source. The two distances are given by

$$r_1 = \sqrt{\left(y - \frac{d}{2}\right)^2 + L^2} \quad \text{and} \quad r_2 = \sqrt{\left(y + \frac{d}{2}\right)^2 + L^2} \quad \text{so when we add the two fields}$$

together to get the total field at the screen we get the kind of mess Maple was invented to handle:

>

Define functions for r1(y) and r2(y)

> **R1:=y->sqrt((y-d/2)^2+L^2);**

> **R2:=y->sqrt((y+d/2)^2+L^2);**

$$R1 := y \rightarrow \sqrt{\left(y - \frac{1}{2}d\right)^2 + L^2}$$

$$R2 := y \rightarrow \sqrt{\left(y + \frac{1}{2}d\right)^2 + L^2}$$

The straightforward thing to do now would just be to add the wavefunctions from the two sources, take the real part, and square it to get the intensity, like this:

> **Intensity:= y-> Re(exp(I*k*R1(y)-I*omega*t)/R1(y) + exp(I*k*R2(y)-I*omega*t)/R2(y))^2;**

$$Intensity := y \rightarrow \Re \left(\frac{e^{(IkR1(y) - I\omega t)}}{R1(y)} + \frac{e^{(IkR2(y) - I\omega t)}}{R2(y)} \right)^2$$

>

Unfortunately, it's not that simple. The problem is with the time. If we just choose some convenient time, like $t=0$, then we will be comparing instantaneous intensities at different places and at each place the intensity varies between zero and some maximum. We don't want to see fake zeros in our intensity pattern on the screen just because we look at a special time. To get rid of this problem we want to find

the *time-averaged intensity* .

$$\int_0^T f(t) dt$$

The time average of an oscillating quantity is defined this way: $\langle f(t) \rangle = \frac{\int_0^T f(t) dt}{T}$, where T is the

period of the oscillation. This looks a little complicated, but is really just a dressed-up version of what you normally think of as the average: (i) add up all the data points (integration is addition of a whole bunch of $f \cdot dt$'s) and (ii) divide by the number of points (divide by the integration interval). The connection works like this

$$\frac{\int_0^T f(t) dt}{T} = \frac{\sum f(t_i) dt}{T} \quad \text{but} \quad \frac{T}{dt} = N \quad , \text{ the approximate number of data points, so we have}$$

$$\frac{\int_0^T f(t) dt}{T} = \frac{\sum f(t_i)}{N} \quad , \text{ the usual average.}$$

Since we usually work with sines and cosines it is useful to know what the time averages of sines, cosines, and their products are:

> **Int(sin(t),t=0..2*Pi)/(2*Pi);value(%);**

$$\frac{1}{2} \int_0^{2\pi} \sin(t) dt$$

0

>

> **Int(cos(t),t=0..2*Pi)/(2*Pi);value(%);**

$$\frac{1}{2} \int_0^{2\pi} \cos(t) dt$$

0

>

> **Int(cos(t)*sin(t),t=0..2*Pi)/(2*Pi);**

$$\frac{1}{2} \int_0^{2\pi} \cos(t) \sin(t) dt$$

> **value(%);**

0

> **Int(cos(t)^2,t=0..2*Pi)/(2*Pi);**

$$\frac{1}{2} \int_0^{2\pi} \cos(t)^2 dt$$

> **value(%);**

$$\frac{1}{2}$$

> **Int(sin(t)^2,t=0..2*Pi)/(2*Pi);value(%);**

$$\frac{1}{2} \int_0^{2\pi} \sin(t)^2 dt$$

$$\frac{1}{2}$$

> **value(%);**

$$\frac{1}{2}$$

You need to memorize these results: the time averages of $\sin(t)$, $\cos(t)$, and

$\sin(t)\cos(t)$ are all zero. The time averages of $\cos(t)^2$ and $\sin(t)^2$ are both 1/2. These

answers even make sense: sin, cos, and sin*cos all spend equal amounts of time being positive and negative, so they average to zero. Their squares oscillate symmetrically between 0 and 1, so their averages are 1/2.

OK, having learned this we can tackle the intensity again. Here's what Maple gives us for the intensity in

expression form:

> **assume(k,real,r1,real,r2,real,omega,real,t,real);**

> **Intense:= Re(exp(I*k*r1-I*omega*t)/r1 + exp(I*k*r2-I*omega*t)/r2)^2;**

$$Intense := \left(\frac{\cos(k r_1 - \omega t)}{r_1} + \frac{\cos(k r_2 - \omega t)}{r_2} \right)^2$$

To make the calculation simpler, let's redefine the arguments of these cosine functions to be x1 and x2, respectively:

> **Intense:=(cos(x1)/r1+cos(x2)/r2)^2;**

$$Intense := \left(\frac{\cos(x_1)}{r_1} + \frac{\cos(x_2)}{r_2} \right)^2$$

Now expand it like this

> **Intense:=expand(Intense);**

and then time average each term.

$$Intense := \frac{\cos(x_1)^2}{r_1^2} + \frac{2 \cos(x_1) \cos(x_2)}{r_1 r_2} + \frac{\cos(x_2)^2}{r_2^2}$$

The two squared terms are easy: both x1 and x2 have ωt in them so they just time average to 1/2 (the phase shift due to $k r_i$ only shifts the function; it still spends equal amounts of time between 0 and 1.) The mixed term we have to handle with a trig identity:

> **combine(cos(x1)*cos(x2));**

$$\frac{1}{2} \cos(x_1 - x_2) + \frac{1}{2} \cos(x_1 + x_2)$$

(Note: to get trig identities try **expand**, **simplify**, and **combine** .) The x_1+x_2 term goes like $2 \omega t$ and time averages to zero, while the x_1-x_2 term has no time dependence at all, so its time average is just itself. So we get

> **Intense:= 1/(2*r1^2)+cos(x1-x2)/(r1*r2)+1/(2*r2^2);**

$$Intense := \frac{1}{2} \frac{1}{r_1^2} + \frac{\cos(x_1 - x_2)}{r_1 r_2} + \frac{1}{2} \frac{1}{r_2^2}$$

With time averaging complete we can redefine our intensity function of position y

> **Intensity:=y->1/(2*R1(y)^2)+cos(k*R1(y)-k*R2(y))/(R1(y)*R2(y))+1/(2*R2(y)^2);**

$$Intensity := y \rightarrow \frac{1}{2} \frac{1}{R_1(y)^2} + \frac{\cos(k R_1(y) - k R_2(y))}{R_1(y) R_2(y)} + \frac{1}{2} \frac{1}{R_2(y)^2}$$

Now define the details of physical situation with $d=.01\text{mm}$, $L=1\text{m}$, and red light of wavelength 600 nm

> **d:=1e-5;L:=1;k:=2*Pi/600e-9;**

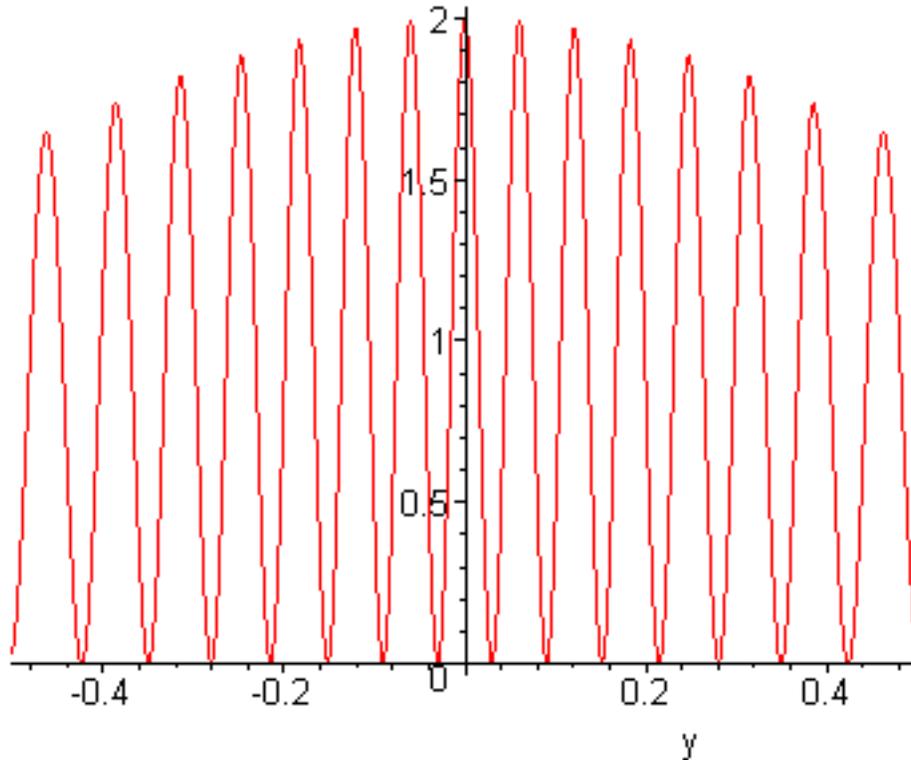
$$d := .00001$$

$$L := 1$$

$$k := .3333333333 \cdot 10^7 \pi$$

Finally, we can plot the intensity as a function of y to see what the pattern should look like on the screen. We will just look in a window 1 m wide near the center of the pattern and see a beautiful interference pattern.

```
> plot(Intensity(y),y=-0.5..0.5);
```



```
>
```

Problem 4.12

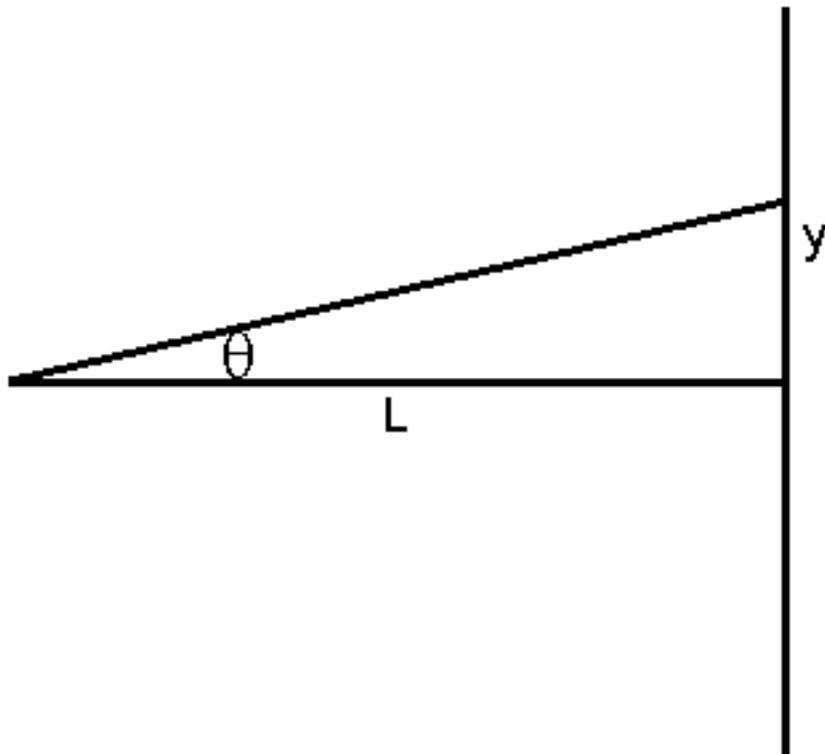
Serway says that the first minimum away from $y=0$ is at the place where $k d \sin(\theta) = \pi$, where θ is the angle indicated below ($y = L \tan(\theta)$). Verify that the first interference minimum in the plot above is at the correct place by measuring y -values on the plot in 4.11 and converting y to angle θ . Use the same values of the variables that were used in Problem/Example 4.11:

```
> restart:with(plots):
```

```
Warning, the name changecoords has been redefined
```

```
> l1:= listplot([[0,0],[2,0]]): # distance to screen
```

```
> t1:= textplot([1,-.1,"L"]): # L label
> l3:= listplot([[2,-1.25],[2,1.25]]): # screen
> t3:= textplot([2.08,.5,"y"]): # y label
> l4:= listplot([[0,0],[2,.6]]): # hypotenuse
> t4:= textplot([.6,.1,"q"]): # theta label
> p1:=plots[display]([l1,t1,l3,t3,l4],axes=none,thickness=3,view=[-.5..2.1,-1.25..1.25],font=[HELVETICA,14]):
> p2:=plots[display]([t4],axes=none,thickness=3,view=[-.5..2.1,-1.25..1.25],font=[SYMBOL,20]):
> display(p1,p2);
```



>

Problem 4.13

Something that Serway doesn't do is show what happens when the screen gets closer and closer to the source. This is complicated because the two fields no longer have about the same magnitude and the distance functions r_1 and r_2 vary more rapidly with y . Rerun the calculation in Problem 4.11 above twice more, once with $L=10d$ and once with $L=2d$ and comment on what these near-field patterns look like. In particular, does the formula given in Problem 4.12 still work? (Be sure to reduce the window in y over which you plot to an appropriate size; $-20d..20d$ is interesting.) There is a lesson here: things are really complicated when you are close to radiation sources.

[Go to top of section](#)

Electrostatics with line charges

[Debugging](#)

The complex plane is a wonderful place to visualize electric fields and equipotential surfaces. Since the complex plane is only two-dimensional, the only kinds of fields you can visualize there are the fields of infinitely long line charges. It turns out (you will just have to trust me until you take a course in complex

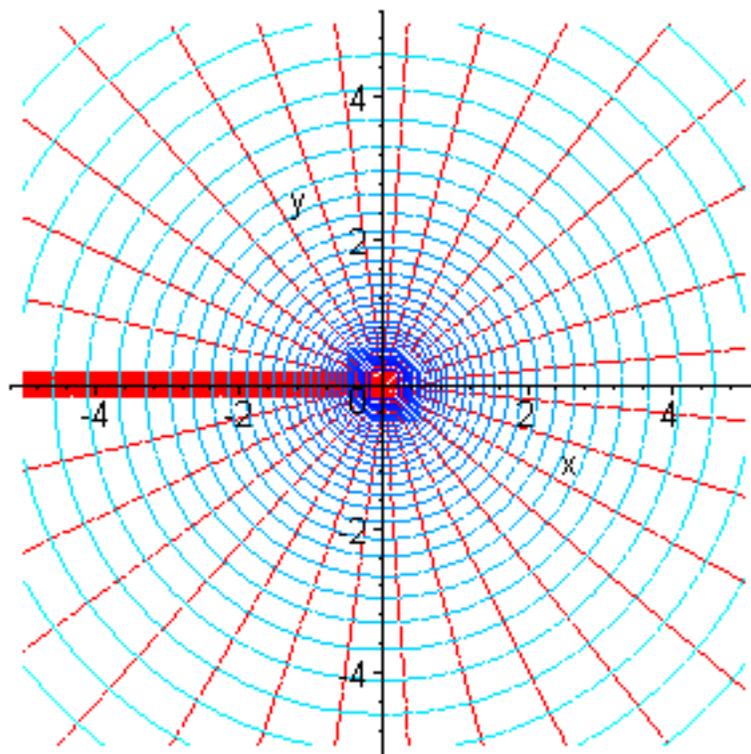
analysis) that the complex function $F(z) = -\frac{\lambda \ln(z - z_0)}{2\pi\epsilon_0}$ has the property that its real part is the

electrostatic potential of a line charge with charge/length λ located in the xy -plane at complex

position z_0 , and that the lines in the complex plane where the imaginary part is constant are the electric field lines! This function combined with Maple's plotting ability makes it possible to quickly visualize equipotential surfaces and field lines for a variety of line charge arrangements. And in the process we will review how Maple does contour plots ([contour plot](#)). Here is a **contourplot** that shows equipotential surfaces in blue and electric field lines in red for a single line charge. (Since we are just visualizing things I have left off all of the physical constants out front.)

```
> restart:with(plots):
> p1:=contourplot(Re(-ln(x+I*y)),x=-5..5,y=-5..5,grid=[30,30],contours=30,coloring=[cyan,blue]):
> p2:=contourplot(Im(-ln(x+I*y)),x=-5..5,y=-5..5,grid=[30,30],contours=30,coloring=[red,red]):
> plots[display]([p1,p2],scaling=constrained);
```

Warning, the name changecoords has been redefined



This looks quite nice except for the mess in the electric field along the negative x-axis. This is the infamous branch cut of the log function which plots as a cliff. If you can just ignore this lousy feature, the plots look nice. (Another way to visualize the electric field without this annoying cliff is to use Maple's **fieldplot** command, discussed in Chapter 2. A quick way to find it in this book is to click Edit on the toolbar, click Find, then search on fieldplot.)

>

Problem 4.14

Make contour plots just like the one above, but for (a) two line charges of equal density, one at $z = i$ and the other at $z = -i$, (b) two line charges with equal density magnitudes but opposite sign located as in part (a), and (c) put a line charge of strength 2 at $z = -i$ and one of strength 1 at $z = i$. In this part you will need a finer grid and more contours to see what's going on. See if you can identify a place where the electric field is zero.

It is also possible to use complex functions to study the behavior of potential and voltage near

conducting corners. Consider the following contour plot of the function iz^2 .

>

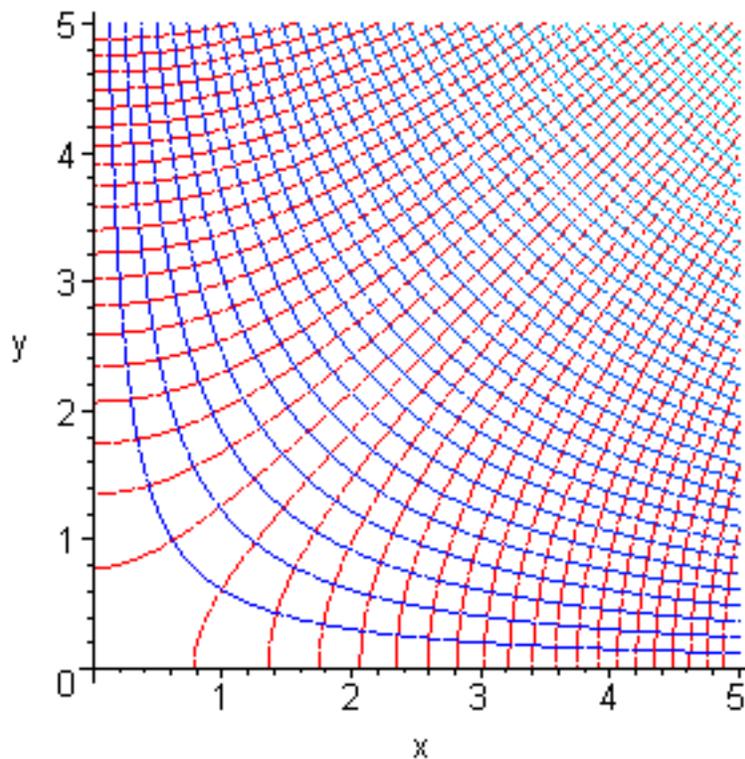
> **restart:with(plots):**

> **p1:=contourplot(Re(I*(x+I*y)^2),x=0..5,y=0..5,grid=[30,30],contours=40,coloring=[cyan,blue]):**

> **p2:=contourplot(Im(I*(x+I*y)^2),x=0..5,y=0..5,grid=[30,30],contours=40,coloring=[red,red]):**

> **plots[display]([p1,p2],scaling=constrained);**

Warning, the name changecoords has been redefined



Recalling that the potential is given by the real part and the field lines by the imaginary part we have

$$iz^2 = -xy + i(x^2 - y^2)$$

and in the plot the blue contours are equipotentials and the red lines are field lines. With $V = -x y$

both the x-axis and the y-axis are grounded, so you are looking at the way electric fields behave in a grounded right-angle corner. You can see the red field lines far apart from each other in the corner, indicating that the electric field is zero there. And how does the electric field behave as we approach the

corner? Remember from electricity and magnetism that $E_x = -\left(\frac{\partial}{\partial x} V\right)$ and $E_y = -\left(\frac{\partial}{\partial y} V\right)$, so

E_x is proportional to y and E_y is proportional to x, so as we approach the corner where $x=0$ and $y=0$, both fields go linearly to zero.

>

We can also find a function that represents the way fields behave in the neighborhood of a convex

$$\left(-\frac{i\pi}{6}\right) \left(\frac{z}{3}\right)$$

corner instead of a concave one by using the function $e^{i\pi/6} z^{2/3}$.

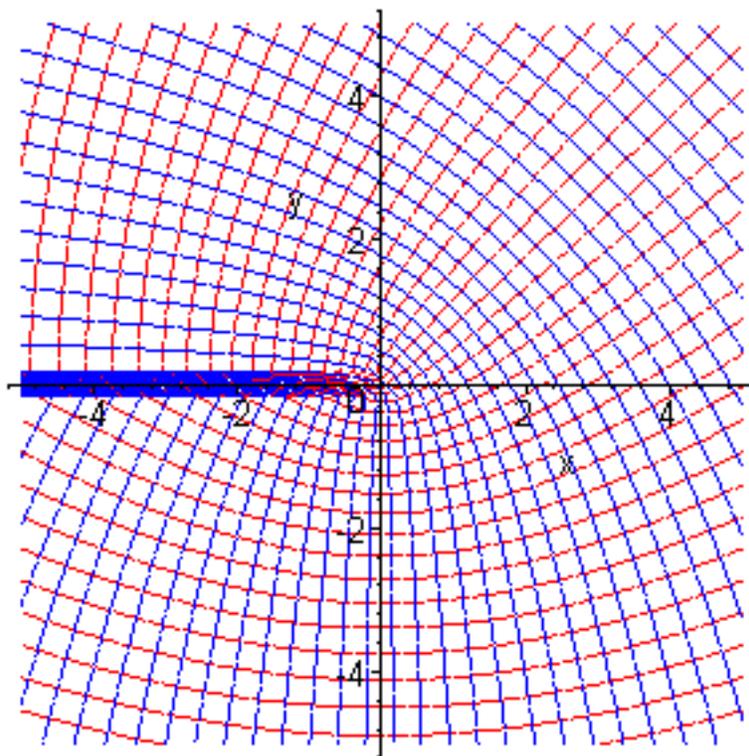
> **restart:with(plots):**

> **p1:=contourplot(Re(exp(-I*Pi/6)*(x+I*y)^(2/3)),x=-5..5,y=-5..5,grid=[30,30],contours=40, coloring=[blue,blue]):**

> **p2:=contourplot(Im(exp(-I*Pi/6)*(x+I*y)^(2/3)),x=-5..5,y=-5..5,grid=[30,30],contours=40, coloring=[red,red]):**

> **plots[display]([p1,p2],scaling=constrained);**

Warning, the name `changecoords` has been redefined



To see how this is the field of a corner, ignore the third quadrant and just focus on quadrants 1, 2, and 4. The blue lines are equipotentials and they get pulled tightly around the corner, and the red field lines

$$\left(\frac{2}{3}\right)$$

radiate out from the corner. In cylindrical coordinates this potential is proportional to $r^{\frac{2}{3}}$ so the

electric field, which goes like the derivative of the potential, is proportional to $\frac{1}{r^{\frac{1}{3}}}$, which goes to

infinity at the corner.

You may be asking yourself where these weird formulas are coming from. They are called conformal mappings and you will study them in a course on complex analysis. It's something fun to look forward to.

>

Problem 4.15

Now I'll let you do a very sharp needle-like corner. The function to use is \sqrt{z} . Your job is to make a picture of potential and E-lines. This time the branch cut will be your friend, because it will be right where the needle is. By thinking about how this potential varies with r in cylindrical coordinates and realizing that the electric field goes like the r-derivative, find out how the electric field blows up as you approach the tip.

[Go to top of section](#)

[Go to top of chapter](#)

[Chapter5.mws](#)

- [Matrices, vectors, and basic operations](#)
 - [Defining matrices and vectors](#)
 - [Add, subtract, and multiply](#)
 - [Inverse, determinant, norm, etc.](#)
 - [Solving systems of linear equations](#)
 - [Vectors: dot and cross products](#)
 - [Eigenvalues and eigenvectors](#)
 - [Vector calculus: grad, div, and curl](#)
-

Chapter 5: Linear algebra

Linear algebra in physics is the art of using matrices and vectors to solve physical problems. Maple knows a lot about linear algebra and you can explore what it knows by typing **?LinearAlgebra**. The sections below just give a brief introduction, but before you start doing any of this stuff you have to tell Maple you want it to use linear algebra, like this:

```
> with(LinearAlgebra):
```

Matrices, vectors, and basic operations

Defining matrices and vectors

[Debugging](#)

The first thing you need to know how to do is to make matrices and vectors. There are multiple ways to do this in Maple, but I suggest you do it in the following way because it is close to the way it is done in Matlab, which is another computer language you are likely to encounter. (It is actually like the HP-48 syntax.) To build a 2x2 matrix A containing 1,2,3,4 you would use

```
> A:=Matrix([[1,2],[3,4]]);
```

$$A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

To refer to the elements of this matrix use the notation **A[row,column]**, i.e. **A[1,2]** like this:

> **A[1,2];**

2

And if you want to just define a 2x2 matrix B but not load values yet you would use

> **B:=Matrix(1..2,1..2);**

$$B := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Once B is defined you can load its elements any way you want, including by hand, like this

> **B[1,1]:=2;B[1,2]:=3;B[2,1]:=7;B[2,2]:=4;**

$$B_{1,1} := 2$$

$$B_{1,2} := 3$$

$$B_{2,1} := 7$$

$$B_{2,2} := 4$$

Another one of Maple's ways of entering matrices is so handy that it should be mentioned here. Click **View** on the toolbar, then select **Palettes**, and then **Matrix Palettes**. You will now be looking at a template for selecting various sizes of matrices. Click on the size you want and the following Maple command will appear, just waiting for you to enter matrix elements in place of the question marks and then to assign it a name with **:=**.

> **Matrix([[%?, %?, %?], [%?, %?, %?], [%?, %?, %?]]);**

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

You might want to look at B to check that it is right, and try this

> **B;**

$$\begin{bmatrix} 2 & 3 \\ 7 & 4 \end{bmatrix}$$

For more details about the **Matrix** command use **?Matrix**.

You can also make column and row vectors using the **Vector** command, like this:

The default is to make a column vector

> **b:=Vector([5,6]);**

$$b := \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

To make a row vector you have to specifically tell Maple, like this

> **b:=Vector[row]([5,6]);**

$$b := [5, 6]$$

And if you want to fill vectors with numbers from the rows or columns of a matrix you can use Maple's **Column** and **Row** commands. For example, to load c with the second column of the matrix A above use

> **c:=Column(A,2);**

$$c := \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

To load c with the first row of A use

> **c:=Row(A,1);**

$$c := [1, 2]$$

There is also a handy short syntax for defining matrices and column vectors. Column vectors can be defined without a special command using the $\langle \dots \rangle$ syntax:

> **`s:=<1,2,3>;`**

$$s := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

And since a matrix can be thought of as a collection of column vectors, matrices can be defined similarly

> **`R:=<<1,2,3>|<4,5,6>|<7,8,10>>;`**

$$R := \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}$$

This short form is sometimes used in the examples in online help so I thought you should know about it.

Since this is Maple, you don't have to just work with numbers. Your matrices and vectors can also be formulas, like this

> **`d:=Matrix([[x11,x12],[x21,x22]]);`**

$$d := \begin{bmatrix} x11 & x12 \\ x21 & x22 \end{bmatrix}$$

or

> **`e:=Matrix([[x^2*y,y*x],[sin(x),1/x]]);`**

$$e := \begin{bmatrix} x^2 y & y x \\ \sin(x) & \frac{1}{x} \end{bmatrix}$$

Now suppose you want to give x and y values and then evaluate to see what the matrix is

> **eval(e,{x=1,y=2});**

$$\begin{bmatrix} 2 & 2 \\ \sin(1) & 1 \end{bmatrix}$$

Hmmm--apparently not. The problem is that e is not an expression but a matrix of expressions, so Maple needs to use two eval commands: **eval** on the inside to assign values for x and y directly to the elements of e and **evalf** on the outside to force floating point evaluation.

> **evalf(eval(e,{x=1,y=2}));**

$$\begin{bmatrix} 2. & 2. \\ .8414709848 & 1. \end{bmatrix}$$

[Go to top of section](#)

Add, subtract, and multiply

[Debugging](#)

> **restart:with(LinearAlgebra):**

As usual, there are multiple ways of doing these basic operations in Maple, but I am going to show you only one. To find the others use **?LinearAlgebra** To add two 3x3 matrices B and C use the **Add** command.

> **B:=Matrix([[1,2,3],[4,5,6],[7,8,9]]);**

$$B := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

> **C:=Matrix([[3,2,1],[5,6,4],[9,7,8]]);**

$$C := \begin{bmatrix} 3 & 2 & 1 \\ 5 & 6 & 4 \\ 9 & 7 & 8 \end{bmatrix}$$

> **E:=Add(B,C);**

$$E := \begin{bmatrix} 4 & 4 & 4 \\ 9 & 11 & 10 \\ 16 & 15 & 17 \end{bmatrix}$$

Subtraction is just adding with a minus sign

> **E:=Add(B,-C);**

$$E := \begin{bmatrix} -2 & 0 & 2 \\ -1 & -1 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$

You do have to be careful that addition and subtraction make sense. For instance, you can't do this

> **F:=Matrix([[5,7],[1,3]]);**

$$F := \begin{bmatrix} 5 & 7 \\ 1 & 3 \end{bmatrix}$$

> **Add(E,F);**

Error, (in LinearAlgebra:-MatrixAdd) matrix dimensions don't match: 3 x 3 vs 2 x 2

because the matrices are not the same size.

Multiplication of matrices and vectors can be done with the Multiply command:

> **G:=Multiply(B,C);**

$$G := \begin{bmatrix} 40 & 35 & 33 \\ 91 & 80 & 72 \\ 142 & 125 & 111 \end{bmatrix}$$

There is a wordier way to do matrix multiplication which you can find in the command list by going to **?LinearAlgebra**.

Problem 5.1

Verify that Maple did this right by computing G_{22} by hand.

You can also multiply matrices and vectors together, like this

> **b:=<5,6,7>;**

$$b := \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

> **Multiply(B,b);**

$$\begin{bmatrix} 38 \\ 92 \\ 146 \end{bmatrix}$$

As expected, a square matrix times a column vector is a column vector. A row vector times a square matrix would be a new row vector, and Maple knows how to do this too:

> **c:=Vector[row]([1,2,3]);**

$$c := [1, 2, 3]$$

(Or you could use the shorthand form)

> **c:=<1|2|3>;**

$$c := [1, 2, 3]$$

> **Multiply(c,C);**

$$[40, 35, 33]$$

You can also multiply matrices and vectors by scalars using the ordinary multiplication operator

> **3*b;**

$$\begin{bmatrix} 15 \\ 18 \\ 21 \end{bmatrix}$$

> **5*B;**

$$\begin{bmatrix} 5 & 10 & 15 \\ 20 & 25 & 30 \\ 35 & 40 & 45 \end{bmatrix}$$

>

Finally, you will notice that division is not mentioned here. That's because division with matrices and vectors is rather tricky and actually falls under the heading of "Solving systems of linear equations", which comes a little later.

Problem 5.2

An important application of matrix multiplication is the rotation of vectors in three-dimensional space. With a carefully drawn picture and some trigonometry you can show that if a vector \mathbf{v} is rotated by angle θ about the x-axis in the counterclockwise direction when looking from positive x toward the origin, then the new vector is simply given by the matrix \mathbf{R}_x multiplied by \mathbf{v} , where \mathbf{R}_x is given by

> **$\mathbf{R}_x := \text{Matrix}(\llbracket [1,0,0], [0, \cos(\theta), -\sin(\theta)], [0, \sin(\theta), \cos(\theta)] \rrbracket);$**

$$\mathbf{R}_x := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Similarly, the rotation matrices about the y- and z-axes through angles ϕ and ψ are given by

> **$\mathbf{R}_y := \text{Matrix}(\llbracket [\cos(\phi), 0, \sin(\phi)], [0, 1, 0], [-\sin(\phi), 0, \cos(\phi)] \rrbracket);$**

> **$\mathbf{R}_z := \text{Matrix}(\llbracket [\cos(\psi), -\sin(\psi), 0], [\sin(\psi), \cos(\psi), 0], [0, 0, 1] \rrbracket);$**

$$\mathbf{R}_y := \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

$$R_z := \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

>

(Note: there is not a sign mistake on the sines in the R_y matrix. The sine signs are different in this matrix than they are in R_x and R_z because rotation is always defined with respect to the right-hand rule on angle: put your thumb in the direction of the rotation axis and your fingers curl in the direction of positive angle.

Try it with your hand with $\phi = \frac{\pi}{2}$ and check that R_y is correct.) Use these rotation matrices to check where the vector $[1,0,0]$ ends up after the following

sequence of rotations: (a) by $\frac{\pi}{2}$ about the y -axis, then (b) by $\frac{\pi}{4}$ about the x -axis, and finally (c) by $\frac{\pi}{2}$ about the z -axis. After Maple gives you the

answer verify that it is correct by making a coordinate system with your fingers and doing the rotations. (Make sure the coordinate system is right-handed). Then start with the same vector $[1,0,0]$ but do the rotations in the reverse order. Do you end up in the same place? (You don't--we describe this situation by saying that rotation operations don't commute.) Rotation should not change the magnitude of the vector; does this sequence of matrix multiplications preserve the magnitude of the vector? You can get the magnitude of a vector \mathbf{v} with the command **Norm(v,2)**.

[Go to top of section](#)

Inverse, determinant, norm, etc.

[Debugging](#)

There are many other operations you can perform on matrices, and Maple knows how to do a lot of them. To see what's available type **?LinearAlgebra** and browse through the topics. Here are a few of the common ones encountered in physics.

Define some matrices to play with

> **restart:with(LinearAlgebra):**

> **B:=Matrix([[1,2,3],[4,5,6],[7,8,9]]);**

$$B := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

> **C:=Matrix([[3,2,1],[5,6,4],[9,7,8]]);**

$$C := \begin{bmatrix} 3 & 2 & 1 \\ 5 & 6 & 4 \\ 9 & 7 & 8 \end{bmatrix}$$

Matrix Inverse: To get the inverse of a matrix, say C, just do this

> **MatrixInverse(C);**

$$\begin{bmatrix} \frac{20}{33} & \frac{-3}{11} & \frac{2}{33} \\ \frac{-4}{33} & \frac{5}{11} & \frac{-7}{33} \\ \frac{-19}{33} & \frac{-1}{11} & \frac{8}{33} \end{bmatrix}$$

To check it do this

> **Multiply(C , MatrixInverse(C));**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant: Get the determinant of a matrix this way

> **Determinant(B);Determinant(C);**

0

33

Transpose: the transpose of a matrix is obtained by switching rows for columns; think about reflecting the matrix across the main diagonal. It also makes sense to take the transpose of a non-square matrix.

> **Transpose(C);**

$$\begin{bmatrix} 3 & 5 & 9 \\ 2 & 6 & 7 \\ 1 & 4 & 8 \end{bmatrix}$$

> **E:=Matrix([[1,2,3]]);**

$$E := [1 \quad 2 \quad 3]$$

> **Transpose(E);**

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

And in quantum mechanics you will want to take the transpose of the complex conjugate, which is called the **HermitianTranspose** :

> **F:=<<1,I>|<2*I,2>>;**

$$F := \begin{bmatrix} 1 & 2I \\ I & 2 \end{bmatrix}$$

> **HermitianTranspose(F);**

$$\begin{bmatrix} 1 & -I \\ -2I & 2 \end{bmatrix}$$

Trace: The trace of a square matrix is the sum of the diagonal elements.

> **Trace(C);**

17

Norm: The norm of a vector has many definitions. You are used to squaring the elements, adding them together, then taking the square root. We use this definition almost exclusively in physics, but Maple is smarter than this, taking the kth root of the sum of the kth powers of the elements. To use our standard $k=2$ in this definition you would use

> **Norm(E,2);**

$\sqrt{14}$

>

For matrices the norm is more complicated; look under **?LinearAlgebra[Norm]** .

Problem 5.3

Find the inverse of the matrix B defined in this section. Explain why you get an error message instead of an answer.

[Go to top of section](#)

Solving systems of linear equations

[Debugging](#)

Maple knows how to do problems like this:

There are three friends, Alice, Bob, and Chuck. The sum of Alice's and Bob's ages are 1.5 times Chuck's age. Chuck's age minus Bob's age is 6. Alice's age added to Chuck's age is 2.25 as large as Bob's age. If Bob used to date Alice, but is now far away in a foreign country, and if Chuck thinks Alice is really cute, is Bob in big trouble? The mathematical part of this problem is just a problem in linear algebra of the form $A*x = b$, with the ages in the order

[Alice, Bob, Chuck]. Here are A and b:

> **restart;with(LinearAlgebra):**

> **A:=Matrix([[1,1,-3/2],[0,-1,1],[1,-2-1/4,1]]);**

$$A := \begin{bmatrix} 1 & 1 & -\frac{3}{2} \\ 0 & -1 & 1 \\ 1 & -\frac{9}{4} & 1 \end{bmatrix}$$

> **b:=Vector([0,6,0]);**

$$b := \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

We would like to solve for the vector x of the friends' ages. Maple has a command **LinearSolve** that makes this kind of problem really easy:

> **LinearSolve(A,b);**

$$\begin{bmatrix} 19 \\ 20 \\ 26 \end{bmatrix}$$

And that's all there is to it. Any linear system can be solved this way. We will learn later that Maple's equation solving command **solve** can also do problems like this.

>

Problem 5.4

Consider 5 springs with spring constants k_1, k_2, k_3, k_4, k_5 hooked together along the x-axis in this order. If each spring has known relaxed length l_1, l_2, l_3, l_4, l_5 , find the unknown lengths of each spring x_1, x_2, x_3, x_4, x_5 when the springs are stretched until the sum of their lengths is twice their total unstretched length. Have Maple give you the general algebraic solution to this problem (it is moderately horrible), then get a numerical answer using $k_1 = .5, k_2 = 1, k_3 = 2, k_4 = 3, k_5 = 3.5$. Let all the relaxed lengths be the same and equal to 10 cm. Recall that the force exerted by a spring is given by the formula $F = -k(x - l)$. You will also need to remember how to do equilibrium problems from freshman mechanics.

[Go to top of section](#)

Vectors: dot and cross products

[Debugging](#)

Maple knows how to do dot products, cross products, and how to find the angle between two vectors with the commands **DotProduct**, **CrossProduct**, and **VectorAngle**.

```
> restart:with(LinearAlgebra);
```

```
> v1:=<1|2|3>;v2:=<4|5|6>;
```

```
v1 := [1, 2, 3]
```

```
v2 := [4, 5, 6]
```

```
> DotProduct(v1,v2);CrossProduct(v1,v2);
```

```
32
```

```
[-3, 6, -3]
```

That's all there is to it. Oh, and to get the angle between them use

```
> VectorAngle(v1,v2);
```

$$\arccos\left(\frac{16}{539}\sqrt{14}\sqrt{77}\right)$$

or, rather,

```
> evalf(VectorAngle(v1,v2));
```

```
.2257261290
```

Note that it comes out in radians.

```
>
```

[Go to top of section](#)

Eigenvalues and eigenvectors

[Debugging](#)

One of the most important ideas from linear algebra as applied to physics are eigenvalues and eigenvectors. They show up in classical mechanics, math physics, quantum mechanics, numerical analysis, and other places. So heads up here, this is important.

The eigenvalue problem as applied to matrices is this: given a square matrix \mathbf{A} , what are the vectors x_i and numbers λ_i such that when \mathbf{A} is multiplied into x_i the result is the same vector x_i , but multiplied by λ_i , or in other words, such that $\mathbf{A} x_i = \lambda_i x_i$.

Since matrix multiplication by a vector simultaneously rotates the vector and multiplies it by a factor, it is not clear that there is any solution at all to this problem. But it can be proven mathematically that in most cases, if the matrix is $N \times N$, then there are N such vectors and values (called eigenvectors and eigenvalues). But they may be complex, which sometimes (but not always) means they are unphysical. Here's an example.

```
> restart:with(LinearAlgebra);
```

```
> A:=Matrix([[1,2,3],[4,3,2],[-1,0,2]]);
```

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

> **Eigenvectors(A);**

$$\left[\begin{array}{l} \frac{1}{2}(-4+4I\sqrt{31}) \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{4}{(-4+4I\sqrt{31})} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 \\ -\frac{1}{4}(-4+4I\sqrt{31}) \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{2}{(-4+4I\sqrt{31})} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 + \frac{1}{2}I\sqrt{3} \left(\frac{1}{2}(-4+4I\sqrt{31}) \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{4}{(-4+4I\sqrt{31})} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right) \\ -\frac{1}{4}(-4+4I\sqrt{31}) \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{2}{(-4+4I\sqrt{31})} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 - \frac{1}{2}I\sqrt{3} \left(\frac{1}{2}(-4+4I\sqrt{31}) \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{4}{(-4+4I\sqrt{31})} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right) \\ -\frac{1}{2}(-4+4I\sqrt{31}) \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{4}{(-4+4I\sqrt{31})} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \end{array} \right]$$

$$\begin{aligned}
& \frac{1}{4}(-4+4I\sqrt{31})\left(\frac{1}{3}\right) + \frac{2}{(-4+4I\sqrt{31})\left(\frac{1}{3}\right)} - \frac{1}{2}I\sqrt{3}\left[\frac{1}{2}(-4+4I\sqrt{31})\left(\frac{1}{3}\right) - \frac{4}{(-4+4I\sqrt{31})\left(\frac{1}{3}\right)}\right], \\
& \frac{1}{4}(-4+4I\sqrt{31})\left(\frac{1}{3}\right) + \frac{2}{(-4+4I\sqrt{31})\left(\frac{1}{3}\right)} + \frac{1}{2}I\sqrt{3}\left[\frac{1}{2}(-4+4I\sqrt{31})\left(\frac{1}{3}\right) - \frac{4}{(-4+4I\sqrt{31})\left(\frac{1}{3}\right)}\right] \\
& \left[-\frac{1}{2}\left[\frac{1}{2}(-4+4I\sqrt{31})\left(\frac{1}{3}\right) + \frac{4}{(-4+4I\sqrt{31})\left(\frac{1}{3}\right)} + 2\right]^2 + \frac{3}{4}(-4+4I\sqrt{31})\left(\frac{1}{3}\right) + \frac{6}{(-4+4I\sqrt{31})\left(\frac{1}{3}\right)} + \frac{1}{2}\right], \\
& -\frac{1}{2}\left[-\frac{1}{4}(-4+4I\sqrt{31})\left(\frac{1}{3}\right) - \frac{2}{(-4+4I\sqrt{31})\left(\frac{1}{3}\right)} + 2 + \frac{1}{2}I\sqrt{3}\left[\frac{1}{2}(-4+4I\sqrt{31})\left(\frac{1}{3}\right) - \frac{4}{(-4+4I\sqrt{31})\left(\frac{1}{3}\right)}\right]\right]^2 - \frac{3}{8}(-4+4I\sqrt{31})\left(\frac{1}{3}\right) \\
& - \frac{3}{(-4+4I\sqrt{31})\left(\frac{1}{3}\right)} + \frac{1}{2} + \frac{3}{4}I\sqrt{3}\left[\frac{1}{2}(-4+4I\sqrt{31})\left(\frac{1}{3}\right) - \frac{4}{(-4+4I\sqrt{31})\left(\frac{1}{3}\right)}\right],
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \left[-\frac{1}{4}(-4+4I\sqrt{31}) \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{2}{(-4+4I\sqrt{31})} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 - \frac{1}{2}I\sqrt{3} \left[\frac{1}{2}(-4+4I\sqrt{31}) \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{4}{(-4+4I\sqrt{31})} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right] \right]^2 - \frac{3}{8}(-4+4I\sqrt{31}) \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
& - \frac{3}{(-4+4I\sqrt{31})} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{2} - \frac{3}{4}I\sqrt{3} \left[\frac{1}{2}(-4+4I\sqrt{31}) \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{4}{(-4+4I\sqrt{31})} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right] \\
& [1, 1, 1]
\end{aligned}$$

Ok, it's clear we've got to do something about this. We usually just want numbers, not gigantic algebraic expressions. An easy way to get them (for the 142nd time) is to enter the elements of the matrix as floating-point numbers, i.e., they have to have decimal points, like this

> **A:=Matrix([[1.,2.,3.],[4.,3.,2.],[-1.,0.,2.]]);**

$$A := \begin{bmatrix} 1. & 2. & 3. \\ 4. & 3. & 2. \\ -1. & 0. & 2. \end{bmatrix}$$

> **v:=Eigenvectors(A);**

$$v := \begin{bmatrix} 4.36146876618582624 + 0. I \\ -0.528917957294363194 + 0. I \\ 2.16744919110853340 + 0. I \end{bmatrix} \begin{bmatrix} .390945316400000020 + 0. I & -.577404724700000038 + 0. I & .0884894421299999984 + 0. I \\ .9054028790999999946 + 0. I & .7838835172000000046 + 0. I & .8443366790999999987 + 0. I \\ -.1655517625000000006 + 0. I & -.2283208606999999990 + 0. I & -.5284554768000000046 + 0. I \end{bmatrix}$$

Good--much better. Now look at the output. According to online help for **Eigenvectors** the first set of numbers is a column vector containing the eigenvalues, and the second set of numbers is a matrix whose columns are the eigenvectors. So if we wanted to study the third eigenvector we would start by picking off the third eigenvalue like this

> **lambda3:=v[1][3];**

$$\lambda_3 := 2.16744919110853340 + 0. I$$

(I know this looks weird, but since $v[1]$ is a column vector it makes sense to ask for its third element). Now we want the third column of $v[2]$, a job tailor made for Maple's Column command

> **v3:=Column(v[2],3);**

$$v_3 := \begin{bmatrix} .0884894421299999984 + 0. I \\ .8443366790999999987 + 0. I \\ -.5284554768000000046 + 0. I \end{bmatrix}$$

Now we are ready to see if this really is an eigenvector by multiplying v_3 by A . If it works, then $A v_3$ will just give us $\lambda_3 v_3$ back.

> **y:=Multiply(A ,v3);**

$$y := \begin{bmatrix} .1917963699299999821 + 0. I \\ 1.83005685221999981 + 0. I \\ -1.14540039573000008 + 0. I \end{bmatrix}$$

To see if it got it right let's divide y by λ_3 and subtract v_3 from it to see if we get zero

> **y/lambda3-v3;**

$$\begin{bmatrix} .711406211717502401 \cdot 10^{-10} + 0. I \\ -.421340740075493159 \cdot 10^{-11} + 0. I \\ .247857290247566198 \cdot 10^{-10} + 0. I \end{bmatrix}$$

which is exactly right, to the precision of Maple floating point numbers with Digits set to its default value of 10.

>

Problem 5.5

Here's a simple eigen-idea that shows up in numerical analysis all the time. Think about what happens if you pick a vector and just keep multiplying it by A in the example above all the time. Your original vector could be written as a linear combination of the eigenvectors (usually), so you can think about what multiplication by A does to each of the three eigenvector components. Now look, this is not too hard--all that happens is that eventually, the vector with the largest eigenvalue will completely take over because every time you multiply by A you get more of it back than you get of the other two. In the example above the biggest eigenvalue is about twice its nearest competitor, and you already know what repeatedly multiplying by 2 can do. So now do this: choose a random vector; then multiply it by A and divide it by its norm using $\text{Norm}(\mathbf{v},2)$ so that the result is a unit vector; then do this multiplication and renormalization about 20 times. Now find the angle between your final result and the eigenvector with the largest eigenvalue; did your calculation get this eigenvector? Can you explain the angle you get?

Problem 5.6

Go get the rotation matrix R_x from Problem 5.2 and have Maple find its eigenvalues and eigenvectors. Then think about what rotation means and see if you can make physical sense out of Maple's answers. Maple will require you to choose a value for theta, so try

(a) $\theta = 1.3$,

(b) $\theta = \pi$, and

(c) $\theta = \frac{\pi}{2}$.

(Note: you will encounter (c) when you study spin in quantum mechanics.) A good way to do this problem is to define a Maple function $M(\theta)$ that defines the matrix R_x .

> **$M:=\theta \rightarrow \text{Matrix}([[1,0,0],[0,\cos(\theta),-\sin(\theta)],[0,\sin(\theta),\cos(\theta)])];$**

$$M := \theta \rightarrow \text{Matrix}([[1, 0, 0], [0, \cos(\theta), -\sin(\theta)], [0, \sin(\theta), \cos(\theta)]])$$

and then use this function to define the three matrices in parts (a)-(c).

> **?grad**

[Go to top of section](#)

Vector calculus: grad, div, and curl

[Debugging](#)

Maple's ability to do algebra means it can do vector derivatives too using the commands **grad**, **diverge**, **curl**, and **laplacian**. And it can do it using Cartesian, cylindrical, and spherical coordinates, but you have to invoke a different linear algebra package **linalg** to use these commands. Note also that the vectors are defined with a different syntax in this package than in the **LinearAlgebra** package invoked in the earlier parts of this chapter. This package uses row vectors in array form, e.g., [vx,vy,vz].

> **restart:with(linalg):**

Warning, the protected names norm and trace have been redefined and unprotected

Here are some gradient examples in each coordinate system.

Cartesian:

> **f1:=3*x^2+y*sqrt(z);**

> **grad(f1,[x,y,z]);**

$$f1 := 3x^2 + y\sqrt{z}$$

$$\left[6x, \sqrt{z}, \frac{1}{2} \frac{y}{\sqrt{z}} \right]$$

Cylindrical

> **f2:=cos(theta)/sqrt(r^2+z^2);**

> **grad(f2,[r,theta,z],coords=cylindrical);**

$$f2 := \frac{\cos(\theta)}{\sqrt{r^2 + z^2}}$$

$$\left[-\frac{\cos(\theta) r}{(r^2 + z^2)^{\frac{3}{2}}}, -\frac{\sin(\theta)}{r \sqrt{r^2 + z^2}}, -\frac{\cos(\theta) z}{(r^2 + z^2)^{\frac{3}{2}}} \right]$$

Spherical

> **f3:=sin(phi)*r^2*(3*cos(theta)^2-1);**

> **grad(f3,[r,theta,phi],coords=spherical);**

$$f3 := \sin(\phi) r^2 (3 \cos(\theta)^2 - 1)$$

$$\left[2 \sin(\phi) r (3 \cos(\theta)^2 - 1), -6 r \sin(\phi) \cos(\theta) \sin(\theta), \frac{r \cos(\phi) (3 \cos(\theta)^2 - 1)}{\sin(\theta)} \right]$$

>

Maple can do the divergence in all three coordinate systems too. The divergence acts on a vector field, so all three components have to be defined in whatever coordinate system you are in. Here are examples in each coordinate system.

Cartesian

> **g1:=[x^2,y^2,z^2];**

> **diverge(g1,[x,y,z]);**

$$g1 := [x^2, y^2, z^2]$$

$$2x + 2y + 2z$$

Cylindrical

- > **g2:=[r^2*cos(theta),r^2*sin(theta),z^2];**
- > **diverge(g2,[r,theta,z],coords=cylindrical);**

$$g2 := [r^2 \cos(\theta), r^2 \sin(\theta), z^2]$$

$$4r \cos(\theta) + 2z$$

Spherical

- > **g3:=[r^2*cos(theta),r^2*sin(theta),cos(phi)];**
- > **diverge(g3,[r,theta,phi],coords=spherical);**

$$g3 := [r^2 \cos(\theta), r^2 \sin(\theta), \cos(\phi)]$$

$$\frac{6r^2 \cos(\theta) \sin(\theta) - \sin(\phi)}{r \sin(\theta)}$$

>

And Maple can do the curl, like this, using the same vector fields used for the divergence.

Cartesian

- > **curl(g1,[x,y,z]);**

$$[0, 0, 0]$$

Cylindrical

> **curl(g2,[r,theta,z],coords=cylindrical);**

$$[0, 0, 4 r \sin(\theta)]$$

Spherical

> **curl(g3,[r,theta,phi],coords=spherical);**

$$\left[\frac{\cos(\theta) \cos(\phi)}{r \sin(\theta)}, -\frac{\cos(\phi)}{r}, 4 r \sin(\theta) \right]$$

>

And finally, it also knows how to do del-squared, the **laplacian**, in each coordinate system. Here it is for each of the scalar functions used with the gradient.

Cartesian

> **laplacian(f1,[x,y,z]);**

$$6 - \frac{1}{4} \frac{y}{\begin{pmatrix} 3 \\ 2 \end{pmatrix}} z$$

Cylindrical

> **laplacian(f2,[r,theta,z],coords=cylindrical);**

$$\frac{-3 \frac{\cos(\theta) r}{\begin{pmatrix} 3 \\ 2 \end{pmatrix}} + \frac{3 r^3 \cos(\theta)}{\begin{pmatrix} 5 \\ 2 \end{pmatrix}} - \frac{\cos(\theta)}{r \sqrt{r^2 + z^2}} + \frac{3 r \cos(\theta) z^2}{\begin{pmatrix} 5 \\ 2 \end{pmatrix}}}{(r^2 + z^2)} r$$

Spherical

> **laplacian(f3,[r,theta,phi],coords=spherical);**

$$\frac{6 r^2 \sin(\theta) \sin(\phi) (3 \cos(\theta)^2 - 1) - 12 r^2 \sin(\theta) \sin(\phi) \cos(\theta)^2 + 6 r^2 \sin(\theta)^3 \sin(\phi) - \frac{r^2 \sin(\phi) (3 \cos(\theta)^2 - 1)}{\sin(\theta)}}{r^2 \sin(\theta)}$$

>

This capability might be nice, but usually in physics the problem is not calculating these vector derivatives, but in undoing them to find a field. This is much harder and Maple is not much help in this regard.

Problem 5.7

Two of Maxwell's equations for static electric and magnetic fields are $\text{div}(\mathbf{E}) = \frac{\rho}{\epsilon_0}$ and $\text{curl}(\mathbf{B}) = \mu_0 \mathbf{J}$. For each of the following E or B fields use

Maple's vector calculus routines to find formulas for the charge density ρ and the current density \mathbf{J} . Note: \mathbf{E} and \mathbf{B} are vectors, so you need to define them properly to do this problem. For instance, if $B_\theta = k r$ then you need to define \mathbf{B} this way: **$\mathbf{B}:=\text{array}([0,k*\mathbf{r},0]);$**

(a) Spherical coordinates: $E_r = \frac{k}{r^2}$ (point charge) (b) Spherical coordinates: $E_r = k r$ (solid sphere of charge)

(c) Cylindrical coordinates: $E_r = \frac{\lambda}{2 \pi \epsilon_0 r}$ (line charge) (d) Cylindrical coordinates: $E_r = k r$ (solid cylinder of charge)

(e) Cylindrical coordinates: $B_\theta = \frac{k}{r}$ (long wire) (f) Cylindrical coordinates: $B_\theta = k r$ (uniform current in a wire)

(g) Cylindrical coordinates: $B_z = k(a^2 - r^2)$

Problem 5.8

The electrostatic potential of a point dipole is given in spherical coordinates by

$V(r, \theta, \phi) = \frac{p \cos(\theta)}{4 \pi \epsilon_0 r^2}$. Find the electric field of this dipole in spherical coordinates by using the defining relation for the potential $\mathbf{E} = -\text{grad}(V)$.

What is the power law for the way \mathbf{E} falls off with distance from the origin?

[Go to top of section](#)

[Go to top of chapter](#)

>

>

[Chapter6.mws](#)

- [Equations in a single variable](#)
- [Systems of equations](#)

Chapter 6: Solving equations

In the linear algebra chapter you saw how to solve systems of linear equations. These are relatively easy, but Maple can handle much harder non-linear equations as well. In this chapter we will learn how to find the roots of polynomials, solve impossible equations involving both elementary and special functions, and even handle sets of impossible equations.

Equations in a single variable

[Debugging](#)

Maple makes short work of hard equations in a single variable. The commands that do the job are **solve** and **fsolve**, corresponding as usual to symbolic and real number answers. For instance, Maple knows the quadratic formula

> **restart;**

> **s1:=solve(a*x^2+b*x+c=0,x);**

$$s1 := \frac{1}{2} \frac{-b + \sqrt{b^2 - 4ac}}{a}, \frac{1}{2} \frac{-b - \sqrt{b^2 - 4ac}}{a}$$

Notice that the syntax of solve is **solve(equation,variable)**. Both answers are given, separated by commas. And if you give it numerical coefficients, it will try to give you numerical answers

> **s2:=solve(x^2-8*x+14=0,x);**

$$s2 := 4 + \sqrt{2}, 4 - \sqrt{2}$$

Maple even knows the formula for the cubic equation

> **s3:=solve(a*x^3+b*x^2+c*x+d=0,x);**

$$\begin{aligned}
 s3 := & \frac{1}{6} (36 b c a - 108 d a^2 - 8 b^3 \\
 & + 12 \sqrt{3} \sqrt{4 a c^3 - c^2 b^2 - 18 b c a d + 27 d^2 a^2 + 4 d b^3} a)^{\frac{1}{3}} \\
 & / a - \frac{2}{3} (3 a c - b^2) / \left(a (36 b c a - 108 d a^2 - 8 b^3 \right. \\
 & + 12 \sqrt{3} \sqrt{4 a c^3 - c^2 b^2 - 18 b c a d + 27 d^2 a^2 + 4 d b^3} a)^{\frac{1}{3}} \\
 & \left. - \frac{1 b}{3 a}, -\frac{1}{12} (36 b c a - 108 d a^2 - 8 b^3 \right. \\
 & + 12 \sqrt{3} \sqrt{4 a c^3 - c^2 b^2 - 18 b c a d + 27 d^2 a^2 + 4 d b^3} a)^{\frac{1}{3}} \\
 & / a + \frac{1}{3} (3 a c - b^2) / \left(a (36 b c a - 108 d a^2 - 8 b^3 \right. \\
 & + 12 \sqrt{3} \sqrt{4 a c^3 - c^2 b^2 - 18 b c a d + 27 d^2 a^2 + 4 d b^3} a)^{\frac{1}{3}} \\
 & \left. - \frac{1 b}{3 a} + \frac{1}{2} I \sqrt{3} \left(\frac{1}{6} (36 b c a - 108 d a^2 - 8 b^3 \right. \right. \\
 & + 12 \sqrt{3} \sqrt{4 a c^3 - c^2 b^2 - 18 b c a d + 27 d^2 a^2 + 4 d b^3} a)^{\frac{1}{3}} \\
 & \left. / a + \frac{2}{3} (3 a c - b^2) / \left(a (36 b c a - 108 d a^2 - 8 b^3 \right. \right. \\
 & + 12 \sqrt{3} \sqrt{4 a c^3 - c^2 b^2 - 18 b c a d + 27 d^2 a^2 + 4 d b^3} a)^{\frac{1}{3}}
 \end{aligned}$$

$$s4 := -\frac{1}{6}(1188 + 12\sqrt{8301})^{\left(\frac{1}{3}\right)} - \frac{10}{(1188 + 12\sqrt{8301})^{\left(\frac{1}{3}\right)}} - 1,$$

$$\frac{1}{12}(1188 + 12\sqrt{8301})^{\left(\frac{1}{3}\right)} + \frac{5}{(1188 + 12\sqrt{8301})^{\left(\frac{1}{3}\right)}} - 1 +$$

$$\frac{1}{2}I\sqrt{3} \left(-\frac{1}{6}(1188 + 12\sqrt{8301})^{\left(\frac{1}{3}\right)} + \frac{10}{(1188 + 12\sqrt{8301})^{\left(\frac{1}{3}\right)}} \right),$$

$$\frac{1}{12}(1188 + 12\sqrt{8301})^{\left(\frac{1}{3}\right)} + \frac{5}{(1188 + 12\sqrt{8301})^{\left(\frac{1}{3}\right)}} - 1 -$$

$$\frac{1}{2}I\sqrt{3} \left(-\frac{1}{6}(1188 + 12\sqrt{8301})^{\left(\frac{1}{3}\right)} + \frac{10}{(1188 + 12\sqrt{8301})^{\left(\frac{1}{3}\right)}} \right)$$

Well, these are numbers correct, but because Maple was thinking symbolically, they don't look very useful. To get actual numbers you can either use our old friend **evalf**

> **evalf(s4);**

-3.953671948, .476835973 - 1.242229238 I,

```
.476835973 + 1.242229238 I
```

or enter the coefficients in the equation with decimal points.

```
> s4:=solve(x^3+3.*x^2-2.*x+7.=0,x);
```

```
s4 := -3.953671947, .4768359736 - 1.242229238 I,
.4768359736 + 1.242229238 I
```

The **solve** command with polynomials is smart: it knows the fundamental theorem of algebra, which is that an n^{th} order polynomial has n roots. For instance if you ask it for the roots of x^6+1 this way

```
> s5:=evalf(solve(x^6+1=0,x));
```

```
s5 := -1. I, 1. I, -.8660254040 - .5000000000 I,
.8660254040 + .5000000000 I, -.8660254040 + .5000000000 I,
.8660254040 - .5000000000 I
```

It will give you all 6. Let's try another one

```
> s6:=solve(x^6+3*x^5+2=0,x);
```

```
s6 := -1,
RootOf(_Z^5 + 2 _Z^4 - 2 _Z^3 + 2 _Z^2 - 2 _Z + 2, index = 1),
RootOf(_Z^5 + 2 _Z^4 - 2 _Z^3 + 2 _Z^2 - 2 _Z + 2, index = 2),
RootOf(_Z^5 + 2 _Z^4 - 2 _Z^3 + 2 _Z^2 - 2 _Z + 2, index = 3),
RootOf(_Z^5 + 2 _Z^4 - 2 _Z^3 + 2 _Z^2 - 2 _Z + 2, index = 4),
RootOf(_Z^5 + 2 _Z^4 - 2 _Z^3 + 2 _Z^2 - 2 _Z + 2, index = 5)
```

You will be seeing **RootOf** a lot as you solve equations in Maple. The problem is that you gave Maple a polynomial with integer coefficients, which tells Maple to try to solve the equation with radicals (square roots, cube roots, etc.). And in this case it couldn't do it, except for the root -1. But if you just want

numbers, either use **evalf**, or change the coefficients to floating point numbers by adding decimal points and you will get all 6.

> **evalf(s6);**

-1., .7266012375 + .4989113292 I, -.2307741868 + .8984935729 I,
 -2.991654101, -.2307741868 - .8984935729 I,
 .7266012375 - .4989113292 I

> **solve(x^6+3.*x^5+2.=0,x);**

-1., -2.991654101, -.2307741868 - .8984935729 I,
 -.2307741868 + .8984935729 I, .7266012375 - .4989113292 I,
 .7266012375 + .4989113292 I

Maple can also handle equations involving harder functions, like this one: $\cos(x) - \frac{x}{10} = 0$.

> **s7:=solve(cos(x)-x/10=0,x);**

$s7 := \text{RootOf}(-_Z + 10 \cos(_Z))$

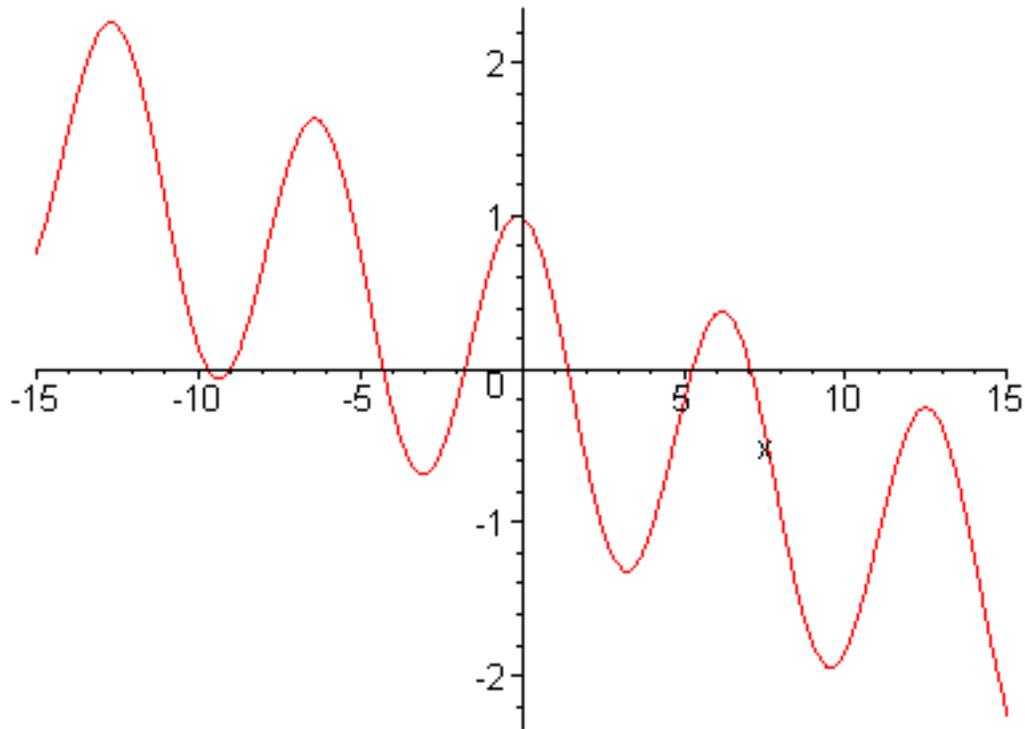
RootOf again, and the solution is the same: change the 10 to 10., or use **evalf(s7)**, to find at least one of the solutions.

> **evalf(s7);**

1.427551779

But when you solve hard equations like this you need to be careful, because there may be more than one answer and Maple will not give you all of them. Let's try plotting the function $\cos(x) - \frac{x}{10}$ and see how many zero-crossings it has to see how many answers we should have found.

> **plot(cos(x)-x/10,x=-15..15);**



>

So Maple didn't actually lie; it just didn't volunteer information. 1.427551779 is a solution, but there are others as well. *This is important: whenever possible, make a plot first so you know how many solutions you are looking for.*

Well, how do we get Maple to give us all of the roots? Since we are only going to get numerical roots anyway we might as well use a numerical solver, which is **fsolve** in Maple. When using **fsolve** you don't just give it an equation and a variable, but rather an equation and a variable which is set equal either to a search range or to a single initial guess for the solution. For instance, in the problem at hand the plot tells us that there are roots near -9.6, -9.2, -4, -2, 1.5, 5, and 7. You must instruct **fsolve** to look near each of these to get them all

```
> fsolve(cos(x)-x/10=0.,x,-10..-9.5);
```

```
-9.678884018
```

```
> fsolve(cos(x)-x/10=0.,x,-9.5..-8.8);
```

```
-8.966016479
```

etc..

Notice what happens if you accidentally specify a range in which there is no solution:

```
> fsolve(cos(x)-x/10=0.,x,-8.5..-8);
```

$$\text{fsolve}\left(\cos(x) - \frac{1}{10}x = 0., x, -8.5 .. -8\right)$$

In general, when Maple just gives back what you gave it, it means either it couldn't find the answer, or you loused up. It's your job to figure out which it is.

Notice also that you can just give it a single guess at where the solution is instead of a range

```
> fsolve(cos(x)-x/10=0.,x=1.5);
```

1.427551779

But be careful here. If you give **fsolve** an initial guess that is close to a maximum of the function whose zeros you are looking for **fsolve** might get lost, or at least go find a solution that is nowhere your initial guess. When you study the secant method in Chapter 8 you will see why this happens.

```
> fsolve(cos(x)-x/10=0.,x=0);
```

7.068891237

Yet another way of controlling which solution Maple gives you is the **avoid** option in **fsolve**. Suppose

you are looking for solutions of the equation $\cos(x) + \frac{x}{3} = 0$. If you plot this function to see where

the zeros are, you will see two closely spaced solutions just below 3. This command will find one of them:

```
> s1:=fsolve(cos(x)+x/3=0,x=3);
```

s1 := 2.938100394

but maybe you wanted the next one down and it is hard to tell from the graph what initial guess or range to use to get it. Well, you can tell Maple to find another one near 3, but not to find s1 again, like this

```
> s2:=fsolve(cos(x)+x/3=0,x=.09,avoid={x=s1});
```

$$s2 := 2.663178883$$

>

Problem 6.1

Find all of the roots, real and complex, of the following polynomials. Use both **solve** and **fsolve** on them. You will notice that **solve** will give you complex values, but **fsolve** won't. You can fix this by telling **fsolve** that you want complex solutions, like this: **fsolve(x^2+1=0,x,complex)** . Also use the command **factor(f,complex)** as well, where f is the polynomial you want to factor.

>

$$(a) x^4 - 1 \quad (b) x^3 + x^2 + x + 1$$

Problem 6.2

Find all of the solutions between -5 and 20 of the equation $e^x = 10 \sin(x)$.

Problem 6.3

Find all of the solutions of the equation $e^{(-x)} = x$. Use both **solve** and **fsolve** and compare their answers.

[Go to top of section](#)

Systems of equations

Debugging

Maple can also handle systems of both linear and nonlinear equations, but this is a much trickier business because it is harder to plot the equations to understand what is going on, and because there is so much more space in which to get lost in multiple dimensions. The commands that handle these systems are **solve** and **fsolve** again, but this time you give them sets (stuff in curly brackets) of equations and variables.

As a first example, let's have **solve** do a simple linear algebra problem like the ones in Chapter 5. John is twice as old as Kimberly. Kimberly's age added to John's age is 27. Find their ages. Using the LinearAlgebra package we would set this up as a matrix, but **solve** (and **fsolve**) can work with the equations directly, like this:

> **restart;**

> **E1:=J=2*K;E2:=K+J=27;**

$$E1 := J = 2 K$$

$$E2 := K + J = 27$$

> **solve({E1,E2},{J,K});**

$$\{J = 18, K = 9\}$$

> **fsolve({E1,E2},{J,K});**

$$\{J = 18, K = 9\}$$

Because you don't have to worry about the conversion to matrix form, this is probably the method of choice for solving systems of linear equations.

But **solve** and **fsolve** can also solve *nonlinear* systems, i.e., system of equations in which the variables are squared, cubed, sit inside sines, cosines, exponentials, etc.. For example, here is a set of two nonlinear equations to solve

> **restart;**

> **E1:=x^2-y=5;E2:=x-y^2=-13;**

$$E1 := x^2 - y = 5$$

$$E2 := x - y^2 = -13$$

Let's try the solve command first

```
> s:=solve({E1,E2},{x,y});
```

```
s := {x = 3, y = 4}, {
x = RootOf(_Z3 + 3 _Z2 - _Z - 4, label = _L1),
y = RootOf(_Z3 + 3 _Z2 - _Z - 4, label = _L1) - 5}
```

Maple was smart; it used equation E1 to eliminate y from equation E2 to get a single equation for x. Then it factored this quartic equation to pull out the answer (x,y)=(3,4) and reduced the rest of the problem to a cubic. If we let **evalf** finish the problem it gives us this

```
> evalf(s);
```

```
{x = 3., y = 4.}, {y = -3.756981175, x = 1.114907541}
```

But if you plot the cubic in RootOf above whose roots are supposed to give the values of x that satisfy the equation, you will see that there are supposed to be two more solutions; where are they? You could try **fsolve**, and you would get this

```
> s:=fsolve({E1,E2},{x,y});
```

```
s := {x = 1.114907541, y = -3.756981174}
```

which is even worse since it only gives us one root. You could look for the roots one at a time by giving **fsolve** initial guesses like this:

```
> s:=fsolve({E1,E2},{x=2.9,y=4.1});
```

```
s := {x = 3.000000000, y = 4.000000000}
```

But there is a way to get Maple to give you all 4 roots. Just repeat the first thing we tried, but with all of the integers replaced with floating point numbers:

> **restart;**

> **E1:=x^2.-y=5.;E2:=x-y^2.=-13.;**

$$E1 := x^2. - y = 5.$$

$$E2 := x - y^2. = -13.$$

> **s:=solve({E1,E2},{x,y});**

$s := \{x = 3., y = 4. \}, \{y = 3.184210129, x = -2.860805853 \},$
 $\{x = -1.254101688, y = -3.427228955 \},$
 $\{x = 1.114907542, y = -3.756981174 \}$

There is a lesson here: be creative and try a few different ways to solve the problem and maybe one of them will work.

>

Problem 6.4

Find all of the solutions (real and complex) of the set of equations

$$y^2 - x^2 = -2$$

$$x - y^3 = 3$$

Here's another nonlinear set to put Maple to the test.

> **restart;**

> **E1:=cos(x)+y-sqrt(z)=-.191748502;E2:=x*y*z=3;E3:=x+y+2*z=8;**

$$E1 := \cos(x) + y - \sqrt{z} = -.191748502$$

$$E2 := x y z = 3$$

$$E3 := x + y + 2 z = 8$$

Let's try **solve** first

> **solve({E1,E2,E3},{x,y,z});**

(Nothing happens.) Well, that was informative. Now let's try **fsolve** with specified ranges for each variable

> **fsolve({E1,E2,E3},{x,y,z},{x=-3..3,y=-3..3,z=-3..3});**

$$\{y = .9999999998, x = 1.000000000, z = 3.000000000\}$$

>

so it looks like $(x,y,z)=(1,1,3)$ is pretty close to a solution. I warn you, however, that in 3 dimensions and higher Maple can just get lost and run forever unless (a) you know there is a solution and (b) you tell Maple approximately where to look.

Problem 6.5

A particle finds itself in a region of space where its potential energy function is given by the formula

$$U(x, y, z) = x^2 + y^4 + z^2 + \cos(12x) \sin(12y) - \cos(12z) \quad . \text{ By using the fact that the force}$$

on the particle is given by $F = -\text{grad}(U)$, find 4 points of equilibrium. (Finding just one will be pretty easy--after that you will have to use the range option in **fsolve** with some creativity to avoid finding the same ones over and over.) To see how to take the gradient in Maple, use

> **?grad**

and rummage through the help page until you can see how to apply it to this problem. Remember: the examples at the bottom are your friends. *Warning: do not use the syntax $-\text{grad}(U)$ in this problem.* If you do, you will get a weird answer back that is hard to deal with. Instead use **grad(-U)** . The problem is subtle and involves how vectors are handled in the **linalg** package. You could get the **-grad(U)** form to work if you used **evalm(-grad(U))** .

[Go to top of section](#)

[Go to top of chapter](#)

>

>

[Chapter7.mws](#)

- [First order differential equations](#)
 - [Second order differential equations](#)
 - [Systems of differential equations](#)
 - [Numerically solving differential equations](#)
 - [How does a differential equation mean?](#)
 - [Nonlinear dynamics and chaos \(advanced topic\)](#)
-

Chapter 7: Ordinary differential equations

Differential equations are the language of physics and Maple can help you solve them. It is a good idea before starting a problem with a differential equation in it to load Maple's full set of differential equation tools, like this

```
> restart;with(DEtools);
```

```
>
```

First order differential equations

[Debugging](#)

```
> restart;with(DEtools);
```

Let's begin by asking Maple to solve the first differential equation you probably ever saw, the equation for radioactive decay. If the decay rate of the atoms in a sample is ξ (g has units of 1/second), then the equation determining the number of atoms still left in the sample is

(Differential equations are built like ordinary equations in Maple, but using the **diff** command:)

```
> E1:=diff(N(t),t)=-g*N(t);
```

$$E1 := \frac{\partial}{\partial t} N(t) = -g N(t)$$

along with the initial condition that at time $t = 0$ there were N_0 atoms in the sample. The problem is to find the function $N(t)$. Maple solves it this way using the command **dsolve**.

```
> dsolve(E1,N(t));
```

$$N(t) = _C1 e^{(-g t)}$$

As expected, the solution is a dying exponential. Maple gives you an unknown constant out front because we didn't tell it what the initial conditions are yet. The constant is not too hard to find because at $t = 0$ the exponential function is 1, so the constant $_C1$ must be N_0 . But **dsolve** will accept initial conditions as well and do even this little bit of work for you. You simply replace the first argument of **dsolve** with a set containing E1 and the initial condition, like this. (See **?dsolve, initial_conditions** for more details)

```
> dsolve({E1,N(0)=No},N(t));
```

$$N(t) = No e^{(-g t)}$$

Ok, now suppose you wanted to get a plot of this solution. First you need to tell Maple what the values of N_0 and g are like this

```
> No := 5;g:=3;
```

$$No := 5$$

$$g := 3$$

and then you do something that seems a little odd. You have to somehow grab the expression on the right-hand side of the solution from **dsolve** so you can pass it into plot. One way to do this would be to start the plot command like this,

```
plot(
```

then stop, use the mouse to highlight the expression on the right-hand side of $N(t)=...$ above and paste it in. But this seems awkward; surely Maple will give you access to the solution it just found? It will, and

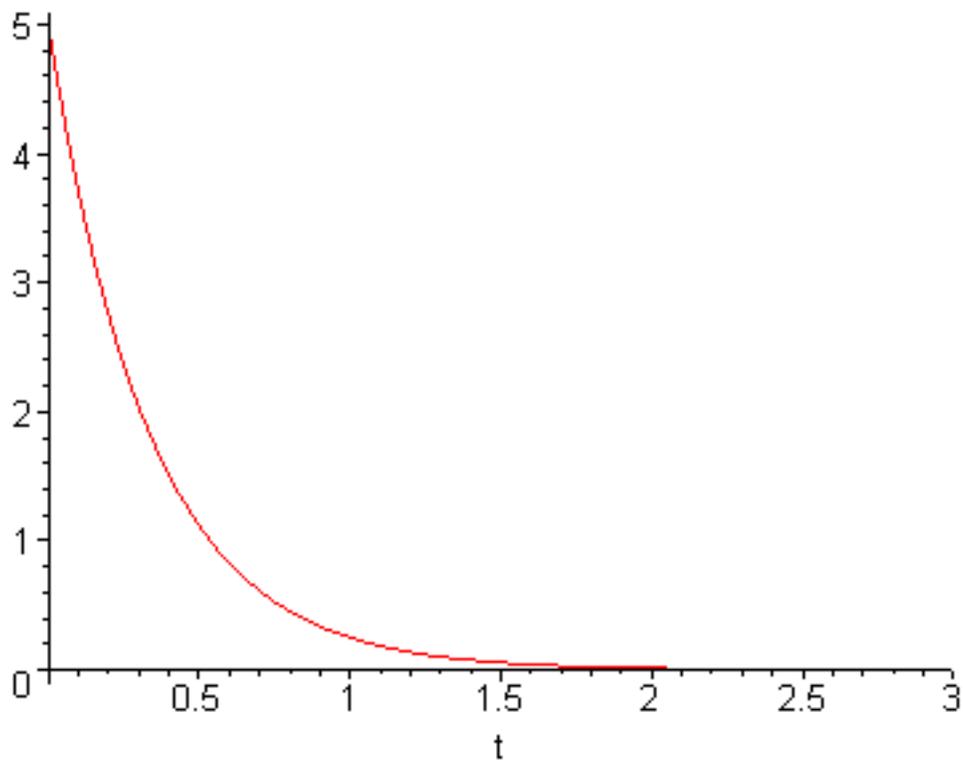
here is one way that works. Instead of just solving the differential equation to the screen, assign the solution to a variable, like this

```
> S1:=dsolve({E1,N(0)=No},N(t));
```

$$S1 := N(t) = 5 e^{(-3 t)}$$

You can now refer to this solution by its name, S1, and in particular you can refer to the solution itself by **rhs(S1)**, like this

```
> plot(rhs(S1),t=0..3);
```



Note that the command **rhs** picks off the right-hand side of an equation. You can probably guess what **lhs** does.

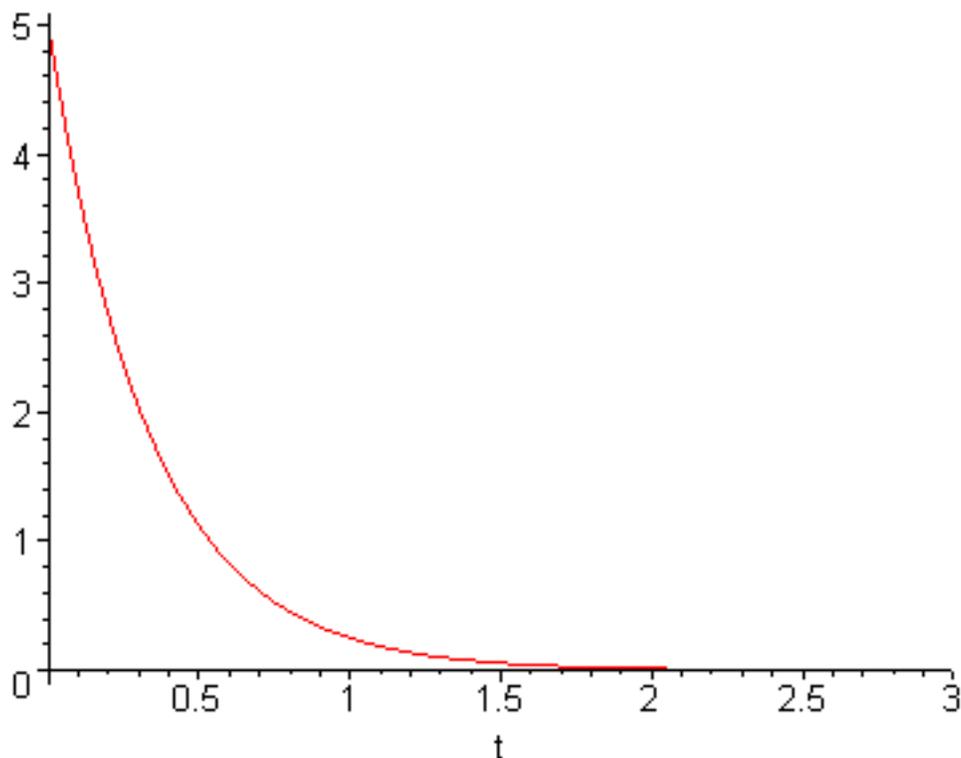
```
>
```

Another way to plot this result is to tell Maple to invent a variable with the name N(t) which can be used as an expression in the plot command. The syntax for this is simply **assign(S1);**.

```
> assign(S1);
```

and then we can use

```
> plot(N(t),t=0..3);
```



The question naturally arises, "What exactly has become of N ?" If you try to look at it this way

```
> N;
```

N

there isn't anything there. The problem is that Maple did something perfectly natural for it, but rather odd for us. It took the result for $S1$, which is $N(t) = 5 e^{(-3 t)}$ quite literally and "sort of " made an expression called $N(t)$. Look at

```
> N(t);
```

$5 e^{(-3 t)}$

It makes sense to differentiate this thing

> **diff(N(t),t);**

$$-15 e^{(-3 t)}$$

It makes sense to solve for when it is 1

> **solve(N(t)=1,t);**

$$\frac{1}{3} \ln(5)$$

But you can easily see that it isn't a function that you can plug numbers into by doing this:

> **N(5);**

$$N(5)$$

And if you try to give t a value and then evaluate the expression N(t), not only does it not work, but it ruins N(t) as well:

> **t:=3;**

$$t := 3$$

> **evalf(N(t));**

$$N(3)$$

> **solve(N(t)=1,t);**

Error, (in solve) a constant is invalid as a variable, 3

The problem seems to be that the symbol N(t) that comes out of the assign is a funny mix of expression and function, so if you want to use this **assign** syntax instead of **rhs**, and it often does seem to be more natural to do so, be very careful about assigning values to the things that appear in the expression. And also notice that you can't go back to **dsolve** again with different initial conditions either, because N(t) has been ruined--it's time to **restart**.

```
> S1:=dsolve({E1,N(0)=3},N(t));
```

Error, wrong number (or type) of parameters in function diff

```
>
```

This may seem so troublesome that you might never want to use **assign**, but **assign** makes the plot and other commands look so much cleaner than they do with **rhs** that I think it is still worth it sometimes. But don't plan on using it exclusively; **rhs** is often the safer thing to use.

One final note about this: in the plot syntax, in the solve syntax, and in the differentiation syntax when it looks like we are using a Maple *function* N and invoking it with $N(t)$ we are actually using a Maple *expression* (sort of) called $N(t)$. If this is confusing (and it certainly confuses me) it might help to review the difference between expressions and functions way back in Chapter 1. [Expressions and functions](#)

Problem 7.1

The differential equation governing the current $I(t)$ in a circuit containing a battery of emf \mathcal{E} , resistance R , and inductance L is

$$L \left(\frac{\partial}{\partial t} I \right) + R I = \mathcal{E} .$$

Solve this differential equation for the current and plot the result if the initial current is zero, $L = .1\text{e-}2$ H, $R = 100 \Omega$, and $\mathcal{E} = 6$ V.

Problem 7.2

Here is a problem involving the idea of "explosive growth". Suppose that you have discovered some process in which the rate of growth of the quantity \mathcal{Y} is not proportional to \mathcal{Y} itself, as in

exponential growth, but is instead proportional to some power of \mathcal{Y} , say \mathcal{Y}^2 or \mathcal{Y}^3 . Use Maple

to explore these two cases by having it solve the differential equation

$$\frac{\partial}{\partial t} y = y^p$$

for $p = 2$ and $p = 3$ and with the initial condition $y(0) = 1$. Then plot the solution over a big enough time interval that you can see the explosion.

Problem 7.3

Consider the relatively innocuous first-order differential equation

$$\frac{\partial}{\partial x} y = \sin(x y) \quad \text{with } y(0) = 1.$$

Ask Maple to solve it and see what happens. In a later section on solving differential equations numerically we will try to solve this equation again.

[Go to top of section](#)

Second order differential equations

[Debugging](#)

Second order equations are very common in physics because Newton's second law is second order and because diffusion, waves, and quantum mechanics all have governing equations containing second derivatives. Let's start with Newton's law of motion, and just as a warm up, let's do a falling object. The differential equation is just

- > **restart:with(DEtools):**
- > **E2:=diff(y(t),t\$2) = -g;**

$$E2 := \frac{\partial^2}{\partial t^2} y(t) = -g$$

> **S2:=dsolve({E2,y(0)=y[0],D(y)(0)=v[0]},y(t));**

$$S2 := y(t) = -\frac{1}{2} g t^2 + v_0 t + y_0$$

Well, that should be a familiar result. Notice a few things about the way this was done. To enter an initial condition on the derivative of y use **D(y)**, which means dy/dt (**D** is Maple's generic derivative symbol) then tack on **(0)** to tell it that the derivative is evaluated at t=0. Yes, this means that **diff** and **D** both mean derivative, and there is an important and subtle difference between them that is hard to remember; it is discussed in the section on derivatives in Chapter 3 ([diff](#) and [D](#)). So either memorize the syntax of the **dsolve** command above, or remember where it is in this document so you can easily find it when you need to solve problems like this.

Here's another famous second order differential equation, the harmonic oscillator.

> **E3:=diff(y(t),t\$2)=-omega^2*y(t);**

$$E3 := \frac{\partial^2}{\partial t^2} y(t) = -\omega^2 y(t)$$

Maple solves it in its generic form with arbitrary constants this way

> **dsolve(E3,y(t));**

$$y(t) = _C1 \sin(\omega t) + _C2 \cos(\omega t)$$

Warning: this form of the solution is dangerous because there is no guarantee that every time you run a worksheet with this **dsolve** command in it that **_C1** will go with sine and **_C2** will go with cosine. Maple has its own internal logic about how to make this choice, but it can turn out to look pretty random in more complicated differential equations than this one. So don't let Maple make the choice. Once you get this form of the solution, use the mouse to copy the solution onto a new line and choose your own unknown coefficients like this (and get rid of the awkward y(t) = and replace it with a genuine assignment as well:)

```
> ysol:= A*sin(omega*t)+B*cos(omega*t);
```

$$ysol := A \sin(\omega t) + B \cos(\omega t)$$

Any calculations you do to determine A and B will now work the same way every time because you have determined for all time that A goes with sine and B goes with cosine.

If the constants are to be determined by initial conditions there is a form of the **dsolve** command that will determine them automatically. For instance, suppose that you want to solve the differential equation E3 with initial conditions $y(0)=1$ and $dy/dt(0)=2$. Then you can use

```
> s:=dsolve({E3,y(0)=1,D(y)(0)=2},y(t));
```

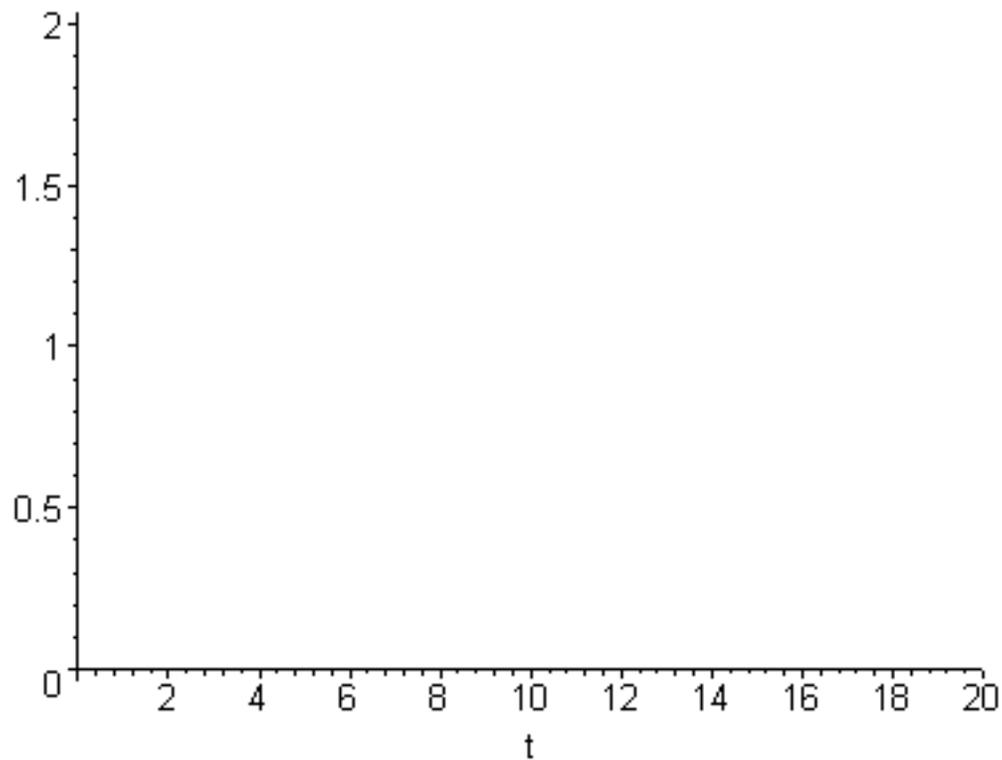
$$s := y(t) = 2 \frac{\sin(\omega t)}{\omega} + \cos(\omega t)$$

Now we have Maple give us the expression $y(t)$ by using **assign**

```
> assign(s);
```

And then we can plot the solution

```
> plot(y(t),t=0..20);
```



Hmmm; that didn't work too well. The plot frame is there, but no function. And, unfortunately, no hint about what went wrong. To debug this problem it would be nice to put some values for t into $y(t)$, but remember that $y(t)$ looks like an expression, but isn't quite, and that assigning t a value will ruin it. So let's just look at $y(t)$ this way and see if we can see what's wrong

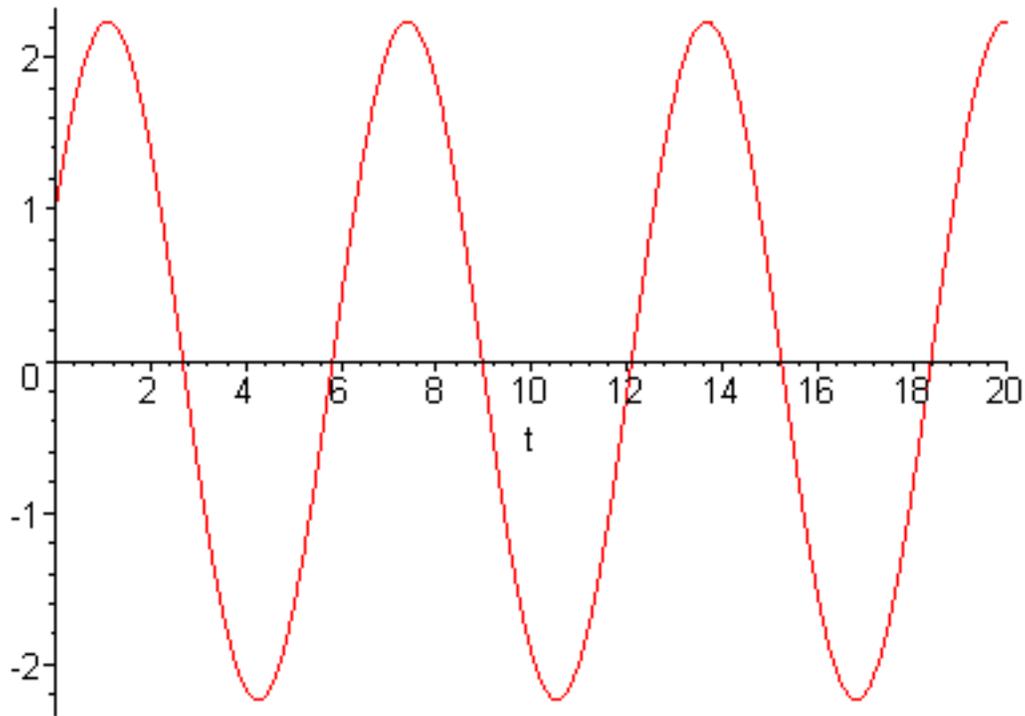
> **y(t);**

$$2 \frac{\sin(\omega t)}{\omega} + \cos(\omega t)$$

The plot command will supply numerical values for t , but the fact that ω is just sitting in this expression without having a numerical value tells us what is wrong; we haven't given ω a numerical value! Ok, so now do this

> **omega:=1;plot(y(t),t=0..20);**

$$\omega := 1$$



Now this problem was simple enough that you could have easily done it yourself once you had taken a class about differential equations. But Maple can do harder problems with ease as well. Let's add some

damping to the harmonic oscillator by putting in a damping force of the form $F = -\frac{m v}{\tau}$, where

τ is the characteristic time for damping to occur. The new equation of motion is now

> **restart:with(DEtools):**

> **Eq:=diff(y(t),t\$2) + diff(y(t),t)/tau + omega^2*y(t)=0;**

$$Eq := \left(\frac{\partial^2}{\partial t^2} y(t) \right) + \frac{\frac{\partial}{\partial t} y(t)}{\tau} + \omega^2 y(t) = 0$$

and we get the general solution this way

> **dsolve(Eq,y(t));**

$$y(t) = C_1 e^{\left(\frac{1(-1 + \sqrt{1 - 4\tau^2\omega^2})t}{2\tau} \right)} + C_2 e^{\left(-\frac{1(1 + \sqrt{1 - 4\tau^2\omega^2})t}{2\tau} \right)}$$

>

Well, that looks impressive. But if it's supposed to be an oscillator that damps, where are the sines and cosines? The problem is that Maple doesn't yet know how big ω and τ are. Let's think physically

for a minute. Suppose we have a natural frequency ω of about 2π , so the period $T = \frac{2\pi}{\omega}$ is 1

second. Now let's suppose that we have this oscillator (think of a pendulum) in motor oil at 50 degrees below zero so that the characteristic damping time is 0.05 seconds (so much friction that motion stops in 0.05 seconds). Your intuition should tell you that if you pull the pendulum back and release it, you are not going to see any swinging; all that is going to happen is that the pendulum will slowly ooze its way into the vertical position and stay there. Let's see what Maple says

> **omega:=2*Pi;tau:=.05;**

$$\omega := 2\pi$$

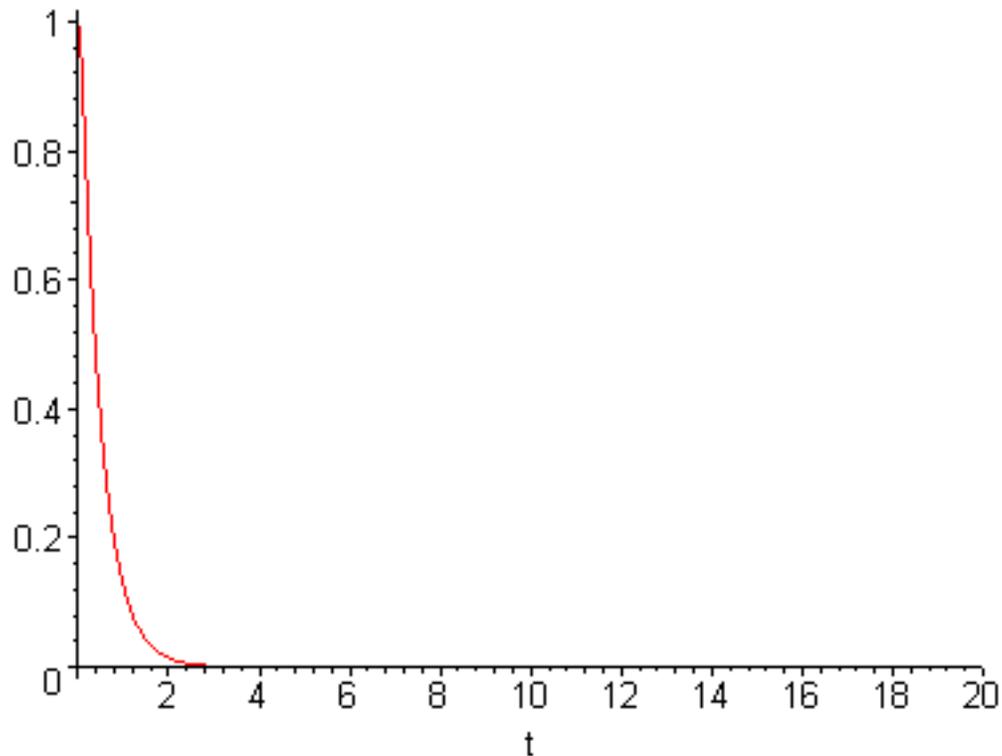
$$\tau := .05$$

> **Y:=dsolve({Eq,y(0)=1,D(y)(0)=0},y(t));**

$$Y := y(t) = \frac{1(5 + \sqrt{25 - \pi^2}) e^{(2(-5 + \sqrt{25 - \pi^2})t)} + 1(5 - \sqrt{25 - \pi^2}) e^{(2(-5 - \sqrt{25 - \pi^2})t)}}{2\sqrt{25 - \pi^2}}$$

$$+ \frac{\frac{1}{2}(-5 + \sqrt{25 - \pi^2}) e^{(-2(5 + \sqrt{25 - \pi^2})t)}}{\sqrt{25 - \pi^2}}$$

> **plot(rhs(Y),t=0..20);**



Ok, Maple agrees with your intuition.

>

Now let's change the damping and imagine what will happen. Your intuition should tell you that if you warm the oil, or use WD-40 instead, or maybe even just air, the pendulum will start to swing and only damp slowly. For example, if $\tau = 20$ then Maple gives us

> **omega:=2*Pi;tau:=20;**

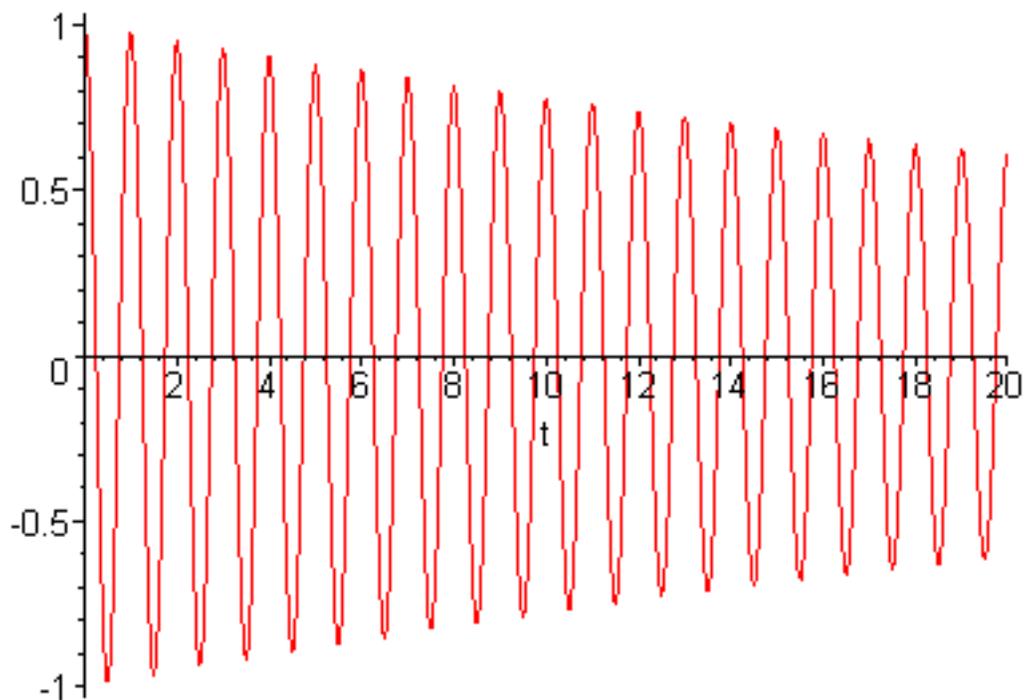
$$\omega := 2\pi$$

$$\tau := 20$$

> **Y:=dsolve({Eq,y(0)=1,D(y)(0)=0},y(t));**

$$Y := y(t) = \frac{e^{\left(-\frac{1}{40}t\right)} \sin\left(\frac{1}{40}\sqrt{-1 + 6400\pi^2}t\right)}{\sqrt{-1 + 6400\pi^2}} + e^{\left(-\frac{1}{40}t\right)} \cos\left(\frac{1}{40}\sqrt{-1 + 6400\pi^2}t\right)$$

> **plot(rhs(Y),t=0..20);**



Well, now it swings and only slowly damps. By what miracle did the exponential functions in the original solution above become sines and cosines? Remember Euler's formula:

$e^{(i\theta)} = \cos(\theta) + i \sin(\theta)$; in this oscillating case the constants in the exponential become

imaginary, and that's how the formula changes from decay to wiggles.

>

This brings up the question of what value of τ makes the transition from pure damping to damped oscillation. Look back at the general form and you will see that there is a square root

$\sqrt{-(2\tau\omega - 1)(2\tau\omega + 1)} = \sqrt{1 - 4\tau^2\omega^2}$ in the exponentials. If the argument of this square

root is positive, then both of the fundamental solutions in the sum are decaying exponentials and we have damping. If the argument of the square root is negative, then the square root is imaginary, the exponentials are imaginary exponentials, i.e., sines and cosines, and we have oscillation. The transition

between the two is when the argument of the square roots is zero, i.e., when $\tau = \frac{1}{2\omega}$. This special

case is called critical damping.

Problem 7.4

Have Maple solve the damped harmonic oscillator differential equation for the case of critical damping with $\omega = 1$ and plot the function $y(t)$ from $t=0$ to $t=2$ with initial conditions $y(0)=1$ and $dy/dt(0)=0$.

The plot will look just like damping with no hint that you are right on the border between damping and oscillation. To verify that you are right on the border increase τ by 2% and make the plot again.

Window the graph down enough that you can see that $y(t)$ goes a little bit negative, indicating that oscillation has begun. You will have to do some serious windowing to see this effect.

Problem 7.5

One last harmonic oscillator problem:

(a) Let's add a sinusoidal driving force at a frequency a little bit different from the natural frequency ω , say $\omega_d = 1.1\omega$. Have Maple solve this differential equation and plot it from $t=0$ to $t=200$ with $\omega = 1$ and $\tau = 30$. Use completely-at-rest initial conditions, $y(0)=0$ and $dy/dt(0)=0$.

$$\left(\frac{\partial^2}{\partial t^2} y(t) \right) + \frac{\partial}{\partial t} y(t) + \omega^2 y(t) = \sin(\omega_d t)$$

The plot should show the classic pattern of the driven-damped harmonic oscillator: an initial period of start-up with some beating between the natural and driving frequencies, then a transition to a state of oscillation with $y(t)$ matched to the driving frequency. Verify graphically that the final oscillation frequency of $y(t)$ is ω_d .

(b) Study resonance in this system by making plots of $x(t)$ vs. t in the following cases:

(i) $\tau = 30$: $\omega_d = .8 \omega$, $\omega_d = \omega$, and $\omega_d = 1.2 \omega$.

(ii) $\tau = 5$: $\omega_d = .8 \omega$, $\omega_d = \omega$, and $\omega_d = 1.2 \omega$.

(iii) $\tau = .8$: $\omega_d = .8 \omega$, $\omega_d = \omega$, and $\omega_d = 1.2 \omega$.

Verify that as the damping time gets longer, the resonant amplitude gets larger and the resonance gets narrower.

(c) Find formulas for the steady state motion of this driven-damped oscillator by substituting the guess

$$x(t) = A \sin(\omega_d t - \phi)$$

into the differential equation in part (a). Then require the $\cos(\omega_d t)$ and $\sin(\omega_d t)$ in the resulting equation to be separately equal (they are independent functions) to get two equations for the two unknowns A and ϕ . Solve these two equations and verify that the formula for $A(\omega_d)$

agrees with the steady state amplitudes you found in part (b). Finally, make plots of $A(\omega_d)$ and $\phi(\omega_d)$ for the three choices of τ used in part (b) and verify again that a long damping time τ

leads to larger and sharper resonance.

Problem 7.6

Use Maple to solve the following second-order differential equations in general form.

$$(a) \quad x^2 \left(\frac{\partial^2}{\partial x^2} f(x) \right) + x \left(\frac{\partial}{\partial x} f(x) \right) + (x^2 - n^2) f(x) = 0$$

$$(b) \quad x^2 \left(\frac{\partial^2}{\partial x^2} f(x) \right) + x \left(\frac{\partial}{\partial x} f(x) \right) - (x^2 + n^2) f(x) = 0$$

$$(c) \quad (1 - x^2) \left(\frac{\partial^2}{\partial x^2} f(x) \right) - 2x \left(\frac{\partial}{\partial x} f(x) \right) + n(n+1) f(x) = 0$$

[Go to top of section](#)

Systems of differential equations

[Debugging](#)

It is often more convenient to deal with systems of differential equations than with second, third, or higher order differential equations. Indeed, many numerical methods require that you write your differential equation as a system of first order differential equations. Here's a simple example. If we call the acceleration dv/dt and the velocity dx/dt , then the harmonic oscillator equation can be written as the following first order set

$$\frac{\partial}{\partial t} v(t) = -\omega^2 x(t)$$

$$\frac{\partial}{\partial t} \mathbf{x}(t) = \mathbf{v}(t)$$

The Maple command **dsolve** handles systems like this by asking for a set {sets have curly brackets} of equations and initial conditions, a set of functions to be solved for, and perhaps extra arguments, like this

> **restart:with(DEtools):**

> **Eq1:=diff(v(t),t)=-omega^2*x(t);Eq2:=diff(x(t),t)=v(t);**

$$Eq1 := \frac{\partial}{\partial t} v(t) = -\omega^2 x(t)$$

$$Eq2 := \frac{\partial}{\partial t} x(t) = v(t)$$

> **dsolve({Eq1,Eq2,x(0)=1,v(0)=0},{x(t),v(t)});**

$$\{v(t) = -\sin(\omega t) \omega, x(t) = \cos(\omega t)\}$$

Notice that this is actually a nice form to have the answer in because we get both the position and the velocity simultaneously. And another good thing is that you don't have to remember the **D** notation for derivative initial conditions. To plot $x(t)$ and $v(t)$ we first need to give ω a value, then solve again

> **omega:=1;XV:=dsolve({Eq1,Eq2,x(0)=1,v(0)=0},{x(t),v(t)});**

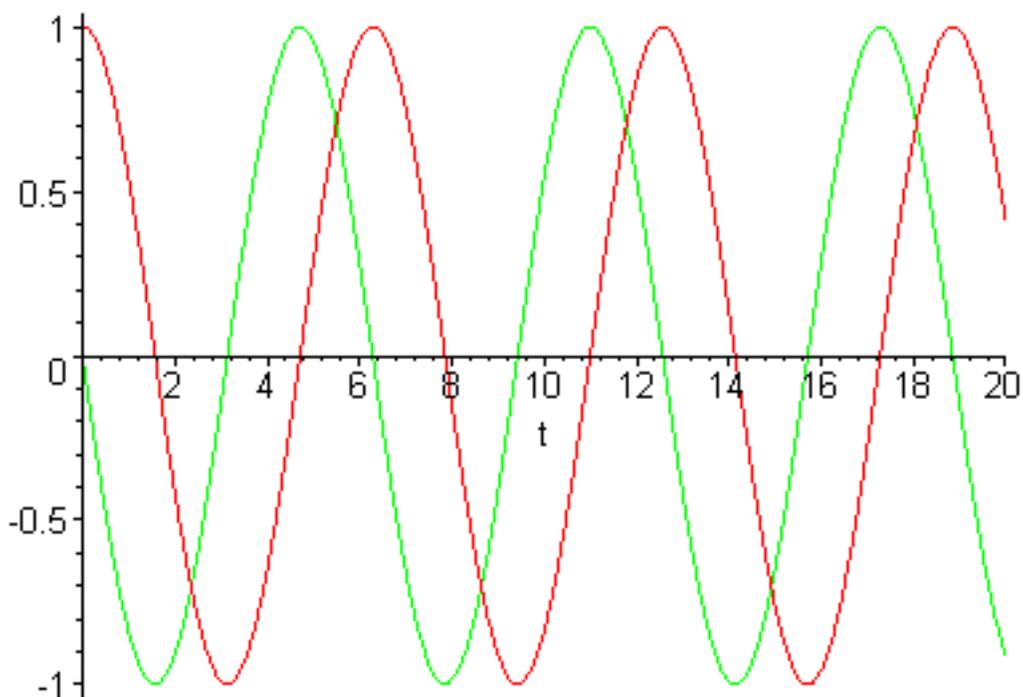
$$\omega := 1$$

$$XV := \{v(t) = -\sin(t), x(t) = \cos(t)\}$$

then make the plots like this using the assign command again

> **assign(XV);**

> **plot([x(t),v(t)],t=0..20);**



>

Problem 7.7

Here are the equations of motion for a charged particle of mass m and charge q moving in the field of a constant magnetic field B in the z -direction and a constant electric field E in the y -direction which come from Newton's second law using the Lorentz force law: $F = q E + q \text{ cross}(v, B)$. (Note: the phrase *equation of motion* has special meaning for physicists and it takes a while for beginning physicists to figure out what we are talking about. It seems natural at first to think that the phrase "equation of

motion" must mean the equations that describe the motion, like $y = y_0 + v_0 t - \frac{g t^2}{2}$, or

$x(t) = A \cos(\omega t)$. This is not what physicists mean when they say "equation of motion". By

equation of motion we mean the differential equation which, when solved, gives the formulas that describe the particle motion. If you can get this straight early on it will save you a lot of trouble.)

$$\frac{\partial}{\partial t} x = v_x \quad \frac{\partial}{\partial t} y = v_y$$

$$\frac{\partial}{\partial t} v_x = \frac{q B v_y}{m} \quad \frac{\partial}{\partial t} v_y = \frac{q E}{m} - \frac{q B v_x}{m}$$

Define these equations in Maple and have Maple solve them with $q=-1.6e-19$, $m=9.11e-31$, $B=0.2$, $E=3000$ with initial conditions

$x(0) = 0$, $y(0) = 0$, $v_x(0) = 20000$, $v_y(0) = 0$. Make plots in the x-y plane of the

particle orbits $x(t)$ and $y(t)$ (they must be parametric plots with t as the parameter). You will need to choose a time interval for plotting and if you just choose something simple like 1 or 2, you will get garbage. Think about the physics of the problem; in particular, think about the cyclotron frequency:

$$\omega_c = \frac{q B}{m} .$$

Also make x-y plane plots of $v_x(t)$ and $v_y(t)$ with the **scaling=constrained**

option. In this way you will be able to see that the velocity goes around in a circle (cyclotron motion) but

with an offset. Verify that the velocity offset (called the "drift velocity") is given by $\frac{\text{cross}(E, B)}{B^2}$.

[Go to top of section](#)

Numerically solving differential equations

[Debugging](#)

Most of the time Maple will not be able to solve your differential equation for you in terms of functions that it knows. When this happens you will want to solve the equation numerically, i.e., perhaps make a plot of the solution, find out what the solution is at $x=2$, etc.. You tell Maple to solve the differential equation numerically by using **dsolve** with the option **type=numeric** , as shown in the example below. But before we start doing this let me warn you. (1) "Numerical" implies that floating point numbers (numbers with decimal points) are used. If you give Maple integers for values in initial conditions or constant definitions, **dsolve(...type=numeric)** may fail. (2) Since the whole point of doing a numerical calculation is to get a number and since Maple is loath to ever give you numbers, doing useful things with what Maple returns as the numerical solution of a differential equation is a bit awkward. You are likely to become more frustrated in this section of the text than any other so far. (3) You will have much better luck at getting Maple to give you what you want if you define your differential equations as first order sets than if you use 2nd, and higher, order derivatives. (4) Assign the result of **dsolve** to a variable name so you can refer to it later, i.e., use **F:=dsolve(...type=numeric)**.

OK, let's start by doing the harmonic oscillator numerically. First define the equations for $x(t)$ and $v(t)$.

> **restart:omega:=1.;**

$$\omega := 1.$$

> **Eqx:=diff(x(t),t)=v(t);Eqv:=diff(v(t),t)=-omega^2*x(t);**

$$Eqx := \frac{\partial}{\partial t} x(t) = v(t)$$

$$Eqv := \frac{\partial}{\partial t} v(t) = -1. x(t)$$

then define the initial conditions

> **IC:=x(0)=1.,v(0)=0.;**

$$IC := x(0) = 1., v(0) = 0.$$

and finally invoke **dsolve** with the numerical option set to get the numerical solution

> **XV:=dsolve({Eqx,Eqv,IC},{x(t),v(t)},type=numeric);**

$$XV := \text{proc}(rkf45_x) \dots \text{end proc}$$

XV is now a procedure that expects to be given a value of t so that it can return an approximate solution to $x(t)$ and $v(t)$, like this

> **XV(2.);**

$$[t = 2., v(t) = -.90929763532898, x(t) = -.41614689959651]$$

Notice that the answer came out apparently accurate to 18 decimal places. In this case we know the answer is just $x(t) = \cos(t)$, so we can check it (after increasing the number of digits Maple keeps in floating point calculations).

> **Digits:=18:cos(2.);**

`-.416146836547142387`

Apparently Maple lied--the result of **dsolve** is pretty accurate (7 decimal places) but not as accurate as the numbers it returned would indicate.

>

We can make it more accurate at the expense of having the solution take longer by specifying the numerical method to use (use **?dsolve,numeric** to see what the choices are; **rkf45** is a pretty good choice; if you have trouble, try another method) and setting the allowable error like this

> **XV:=dsolve({Eqx,Eqv,IC},{x(t),v(t)},type=numeric,method=rkf45,abserr=1e-10);**

`XV := proc(rkf45_x) ... end proc`

> **XV(2);**

`[t = 2., v(t) = -.909297590552880744,
x(t) = -.416146886768602196]`

>

So this is better, but Maple won't let you improve it forever. Try changing the error to 1e-15 in the command above and see what happens. (Don't leave it this small in what follows or you will work your computer to death.)

And now a word of advice: don't try to compute to an accuracy of 1 part in 10000000000000. Five or six significant figures is usually fine, although it is a good idea to do a numerical calculation twice with different error tolerances to make sure the setting you are using most of the time is OK for the problem at hand.

All right, Maple has given us a procedure called **XV** that solves this differential equation numerically. Now suppose we want to plot it. Here's one thing that won't work: you can't use `assign(XV)` like we did with symbolic solutions. The easiest way that does work is to use the Maple command **odeplots** which expects to be given output from **dsolve(...type=numeric)**. Note that you have to use the command **with (plots)** to make this command available.

Here's another problem before we proceed. The error tolerance in the **dsolve** command above is set to $1e-10$, which is so small that asking **odeplot** to find lots of points at this level is asking to be bored in front of a busy screen for a long time. So before we proceed, let's set the error to a more reasonable value for plotting.

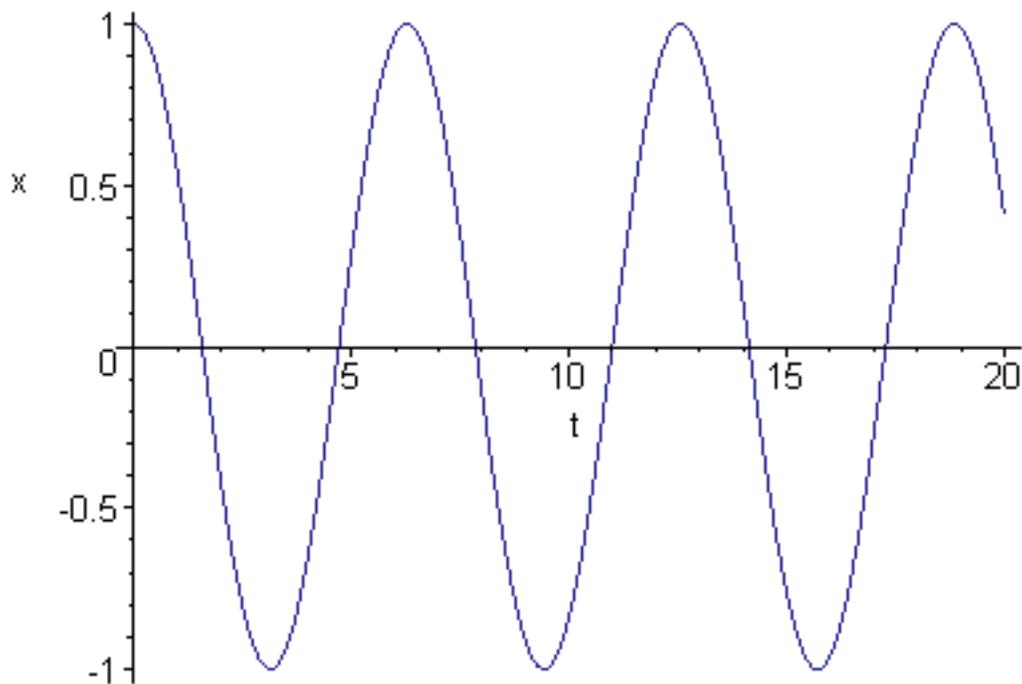
```
> XV:=dsolve({Eqx,Eqv,IC},{x(t),v(t)},type=numeric,method=rkf45,abserr=1e-6);
```

```
XV := proc(rkf45_x) ... end proc
```

In this case it would be used like this to plot $x(t)$

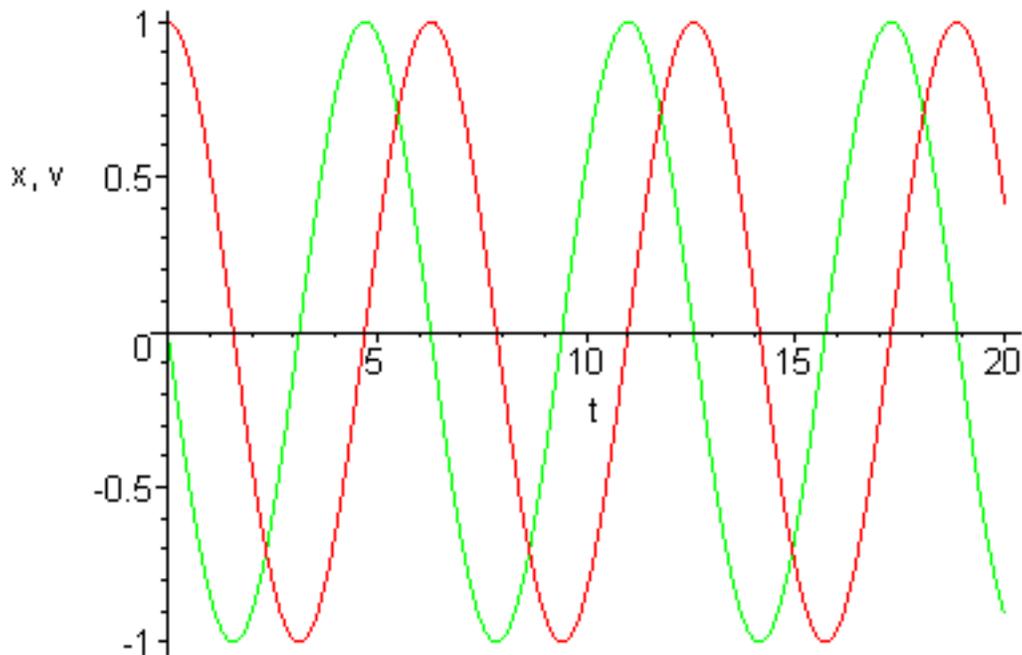
```
> with(plots):odeplot(XV,[t,x(t)],0..20,numpoints=500,color=navy);
```

Warning, the name changecoords has been redefined



or this way to plot both $x(t)$ and $v(t)$

```
> odeplot(XV,[[t,x(t)],[t,v(t)]],0..20,numpoints=200);
```



Note two things. (1) odeplot takes all the usual plot options so you can make it look good and (2) if you have your error set really low and numpoints set really high, you are going to be waiting a long time for your plot to come out.

>

Problem 7.8

Here is problem 7.3 again, which Maple couldn't do symbolically. Do it numerically and plot $y(x)$ from $x=0$ to $x=20$.

$$\frac{\partial}{\partial x} y = \sin(x y) \quad \text{with } y(0) = 1 \quad .$$

Problem 7.9

Here is a real physics problem. In Physics 121 you did the problem of a hard-hit baseball, but because you did it without air friction you were playing baseball on the moon. Let's play ball in a real atmosphere now. The air-friction drag on a baseball is approximately given by the following formula

$$F = - \frac{C_d \rho_{air} \pi r^2 |v| v}{2}$$

where C_d is the drag coefficient (about 0.35 for a baseball), where ρ_{air} is the density of air (about 1.2 kg/m^3), where r is the radius of the ball (about 0.037 m) and where v is the vector velocity

of the ball. The absolute value in this formula pretty much guarantees that Maple won't be giving you a formula for the solution of this problem. If we take the vertical direction to be y and the horizontal direction to be x then Newton's second law in vector-component form gives us the following first-order set of equations

$$\frac{\partial x}{\partial t} = v_x \quad \frac{\partial y}{\partial t} = v_y \quad \frac{\partial v_x}{\partial t} = - \frac{C_d \rho_{air} \pi r^2 v_x \sqrt{v_x^2 + v_y^2}}{2m}$$

$$\frac{\partial v_y}{\partial t} = - \frac{C_d \rho_{air} \pi r^2 v_y \sqrt{v_x^2 + v_y^2}}{2m} - g$$

where m is the mass of the baseball (0.145 kg) and where g is the acceleration of gravity (9.8 m/s^2). Give this problem to Maple by coding the four differential equations and hit a home run with it

using initial conditions

$$x(0) = 0 \quad y(0) = 0 \quad v_x(0) = \frac{v_0}{\sqrt{2}} \quad v_y(0) = \frac{v_0}{\sqrt{2}} \quad \text{where } v_0 = 60 \text{ m/s. Then use } \mathbf{dsolve(}$$

type=numeric) to solve it and make overlaid plots: (i) without air drag (set $C_d = 0$) and (ii) with air drag, as discussed above. Plot $y(t)$ vs. $x(t)$ so you can see the path of the ball by using $[x(t), y(t)]$ in **odeplot** instead of $[t, y(t)]$. Can you see that the trajectory with air friction is not a parabola? Power

hitters in baseball say they would much rather play in Coors Field in Denver than in sea-level stadiums because it is so much easier to hit home runs. Do they know what they are talking about? The air pressure in Denver is about 10% lower than it is at sea level.

Advanced topic: shooting

Problem 7.10

Sometimes we don't want to just solve a differential equation with known initial values. Perhaps there is a parameter in the differential equation that is unknown and we need to determine its value based on the solution of the differential equation at some other point. Or perhaps we want to start out with part of the initial conditions given and then adjust the undetermined initial conditions until that the solution satisfies some other conditions at the far end. Adjusting things so that you start at point a and then hit a target at point b sounds so much like artillery that techniques that solve this kind of problem are usually called *shooting methods*. When a symbolic solution can be found this is not so hard, but when the differential equation must be solved numerically it can be quite difficult. In fact, I can't make Maple do it with its built-in commands, so I will have to give you a procedure based on one written by Dr. Robert Israel to show you how such problems are solved. But first, let's choose a specific problem to solve.

Schrodinger's equation for the bound states of a particle in a harmonic oscillator potential well is this:

$$-\frac{\hbar^2 \left(\frac{\partial^2}{\partial x^2} \psi \right)}{2m} + \frac{kx^2 \psi}{2} = E \psi$$

To make it a nice dimensionless problem we change the length variable from x to ξ by doing this:

$$x = a \xi \quad \text{where } a = \left(\frac{\hbar^2}{2m} \right)^{\frac{1}{4}}$$

and by changing E into the eigenvalue λ by doing this:

$$\lambda = \frac{2 m a^2 E}{\hbar^2} .$$

With these changes of variable Schrodinger's equation becomes

$$\left(\frac{\partial^2}{\partial \xi^2} \psi \right) + (\lambda - \xi^2) \psi = 0 .$$

The solutions of this equation come in two types: even functions in ξ and odd functions in ξ .

Hence we will have two different kinds of initial conditions at $\xi = 0$:

$$\text{even: } \psi(0) = 1 \quad \text{and} \quad \left(\frac{\partial}{\partial \xi} \psi \right)(0) = 0 \quad \text{odd: } \psi(0) = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial \xi} \psi \right)(0) = 1$$

(You may be wondering why we use 1 for the initial value or slope; why not .7, 1e-16, or something else? What we use doesn't matter. If you look at the differential equation every term has ψ in it, so if you double some solution that you have found, the doubled ψ will also be a solution, just twice as big. So since the size doesn't matter we just use 1. If you were going to do real physics, however, then at

the end you would have to choose an overall multiplier for Ψ so that the total probability is 1. You will learn all about this in quantum mechanics.)

The only solutions for the wave function Ψ that make physical sense are the ones that go to zero as ξ approaches ∞ . You will shortly see that this is only true for very special values of the parameter λ , so we need a procedure that does the following: (i) It must be able to solve the differential equation numerically and (ii) it must be able to adjust the parameter λ to make the wavefunction Ψ be small at large ξ .

Well, here is the Maple code containing the shooting procedure, all set up to solve a simpler problem, which is similar to our quantum problem. The practice differential equation is (harmonic oscillator)

$$\left(\frac{\partial^2}{\partial x^2}y\right) + \lambda^2 y = 0 \quad \text{with } y(0) = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial x}y\right)(0) = 1.$$

We want to adjust λ so that we meet the condition $y(1) = 0$. Follow along and run the procedure below, then modify it to find the values of λ in Schrodinger's equation for the first 6 bound states (3 even and 3 odd). You will need to decide how far out to go in ξ to approximate infinity. *Don't overdo it!* Somewhere in the 5-10 range is probably enough.

Start fresh and pull in **odeplot**

```
> restart:with(plots):
```

```
Warning, the name changecoords has been redefined
```

Define the differential equation and assign it to deqn. NOTE: the parameter you are going to vary to satisfy the shooting condition on the right end must be called lambda for this section of code to work.

```
> deqn:=diff(y(x),x$2)+lambda^2*y(x)=0;
```

$$\text{deqn} := \left(\frac{\partial^2}{\partial x^2} y(x) \right) + \lambda^2 y(x) = 0$$

Define the left and right endpoints of the interval

> **a:=0;b:=1;**

$$a := 0$$

$$b := 1$$

Define the initial conditions at the beginning of the interval.

> **bc:=y(a)=0,D(y)(a)=1;**

$$bc := y(0) = 0, D(y)(0) = 1$$

Set the value that you are shooting for, i.e, set the variable yright to the value that y(b) is supposed to be when you have the correct value of the parameter λ

> **yright:=0;**

$$yright := 0$$

Make a guess at the parameter to be varied and plot the numerical solution of the ode from a to b; Adjust the value of λ until you are close to the solution you want. (In the sample problem the parameter values that work are π , 2π , 3π , etc..)

> **lambda:=2.5;**

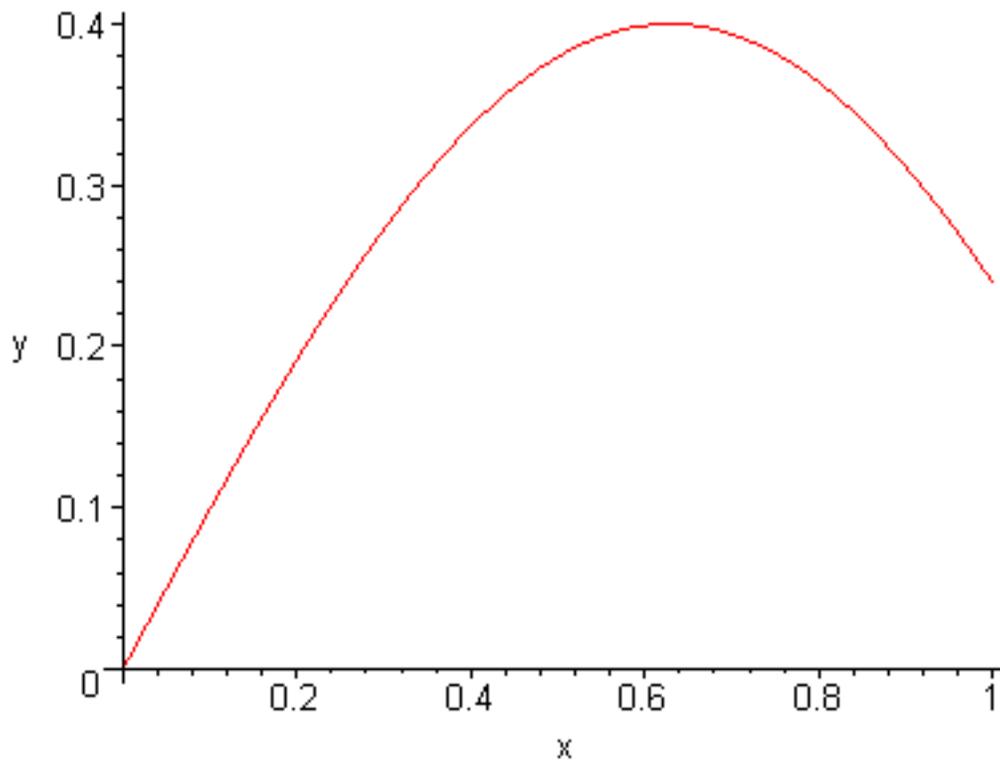
Note that the **startinit=true** option tells **dsolve** to always start with the specified initial conditions. This is important to have set when we use **fsolve** to find the value of param that gives the solution with y(b)=yright.

> **S:=dsolve({deqn,bc},y(x),type=numeric,startinit=true);**

```
> odeplot(S,[x,y(x)],a..b);
```

```
 $\lambda := 2.5$ 
```

```
S := proc(rkf45_x) ... end proc
```



Go back up and try another value of λ until you get close to zero on the right.

I am now going to show you a short section of code that makes it possible to pass the result of **dsolve** into **fsolve** to let Maple find the correct value of λ . The way to make this work was shown to me by Dr. Robert Israel of the University of British Columbia.

```
>
```

Note that this code won't work unless the dummy argument of the procedure F given below (z in this case) is different from all of the variables in the differential equation, i.e., in our problem you can't use y or x in place of z below or you will get in trouble. And because λ has already been defined, you can't use it either; that's why I use the dummy variable z in place of λ . (But why would you want to use x

or y, you ask? In this problem you wouldn't, but in a different problem you might be tempted, and I want you to know that it's a mortal sin.)

```
> F:=proc(z)
> local a,b,deqn,bc,S;
> deqn:=diff(y(x),x$2)+z^2*y(x)=0;
> a:=0;b:=1;
> bc:=y(a)=0,D(y)(a)=1;
> if type(z,numeric) then
> S:=dsolve({deqn,bc},y(x),type=numeric,startinit=true,abserr=1e-6);
> subs(S(b),y(x))
> else
> 'F(z)';
> end if;
> end;
```

```
F := proc(z)
local a, b, deqn, bc, S;
deqn := diff(y(x), x $ 2) + z^2*y(x) = 0;
a := 0;
b := 1;
bc := y(a) = 0, D(y)(a) = 1;
if type(z, numeric) then
S := dsolve( { deqn, bc }, y(x), type = numeric,
startinit = true, abserr = .1*10^(-5));
subs(S(b), y(x))
```

```

else 'F(z)'
end if
end proc

```

$F(\lambda)$ now can be passed into **fsolve** to find a value of λ that satisfies $F(b)=y_{right}$. In the paragraph below you will see how to do this.

Set Digits to a lower value so **fsolve** doesn't kill itself trying to get an answer correct to many significant figures when **dsolve** isn't accurate enough to deliver this much accuracy. If you don't get the right balance between **abserr** in the **dsolve** in the procedure and the **Digits** setting here, **fsolve** will run forever.

```
> Digits:=5;
```

```
Digits := 5
```

```
> F(3.);
```

```
.047040154344257
```

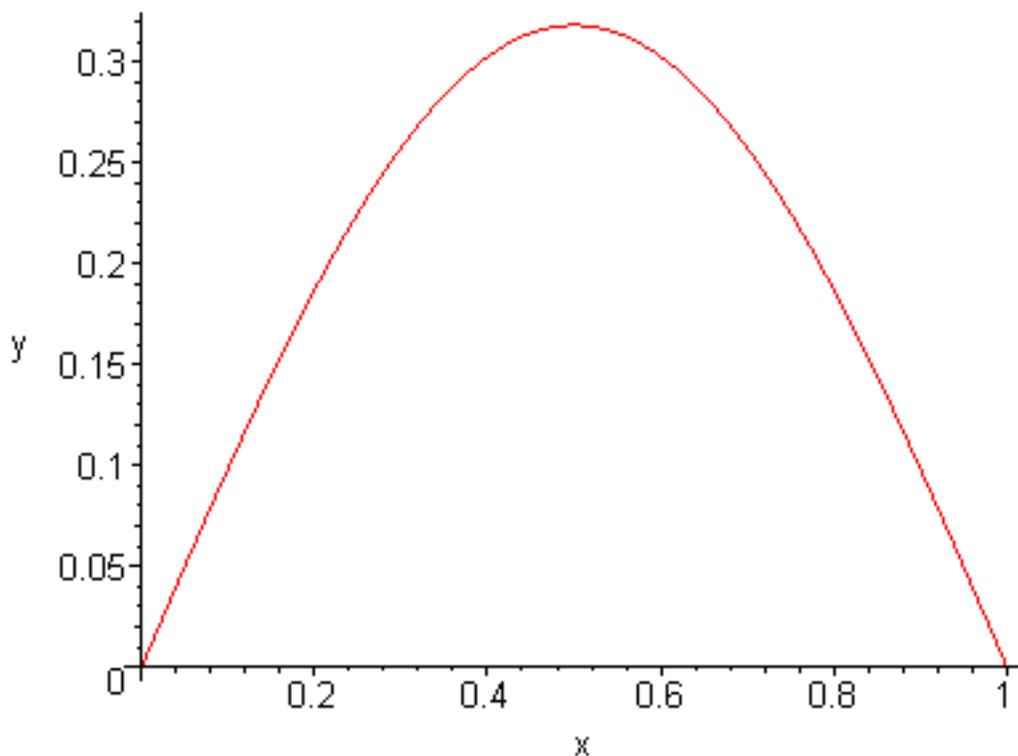
```
> lambda:=fsolve(F(z)=yright,z=3.0..3.2);
```

```
lambda := 3.1416
```

Using this value of λ , make a plot of the solution.

```
> S:=dsolve({deqn,bc},y(x),type=numeric):
```

```
> odeplot(S,[x,y(x)],a..b);
```



Problem 7.11

The equation of motion of a relativistic harmonic oscillator is $\frac{\partial}{\partial t} \mathcal{P} = -m \omega_0^2 x$, where \mathcal{P} is the

relativistic momentum $\mathcal{P} = \frac{m u}{\sqrt{1 - \frac{u^2}{c^2}}}$, where u is the particle velocity $u = \frac{\partial}{\partial t} x$. (a) Write

this equation of motion in the form of two coupled first order differential equations. The first one is easy because it has been given in the first line of this problem. The second one is of the form $\frac{\partial}{\partial t} x = ?$.

Solve for u in terms of \mathcal{P} to get this second equation. (b) Use **`dsolve(...type=numeric)`** to make a

procedure that you can use in odeplot to make graphs of $x(t)$ for the cases $x(0) = .2, .4, 1., 5.$ and $p(0) = 0$. Verify graphically in the last case that the reason that $x(t)$ looks so strange is that it is obeying the relativistic speed limit. In this last case make a plot of the particle speed $u(t)$.

[Go to top of section](#)

How does a differential equation mean?

[Debugging](#)

It is important in physics to have an intuitive feel for what differential equations mean rather than just memorizing a bunch of techniques and formulas. Without this feel you won't be able to propose and refine mathematical models for physical processes. So let's look at a simple one and try to translate it into words:

$$\frac{\partial}{\partial x} y = y$$

Since $\frac{\partial}{\partial x} y$ is the slope of the function $y(x)$ this equation says that the bigger y gets the bigger its

slope gets. Let's try the two possible cases. (i) $y(0)$ is positive. The differential equation then says that the slope is positive, so y is increasing. But if y increases its slope increases, making y increase more, making its slope increase more, etc.. So it is no surprise that the solution of this equation is a function

like e^x that gets huge as x increases. (ii) $y(0)$ is negative. Now the differential equation says that the

slope is negative, so y will have to decrease, i.e., become more negative than it was at $x=0$. But if y is more negative then the slope is more negative, making y even more negative, etc.. Again, it's no surprise

that the solution is a strongly decreasing function like $-e^x$.

With this kind of qualitative analysis in mind, let's go back to an explosive-growth differential equation like the ones you did in the section on first order equations, say

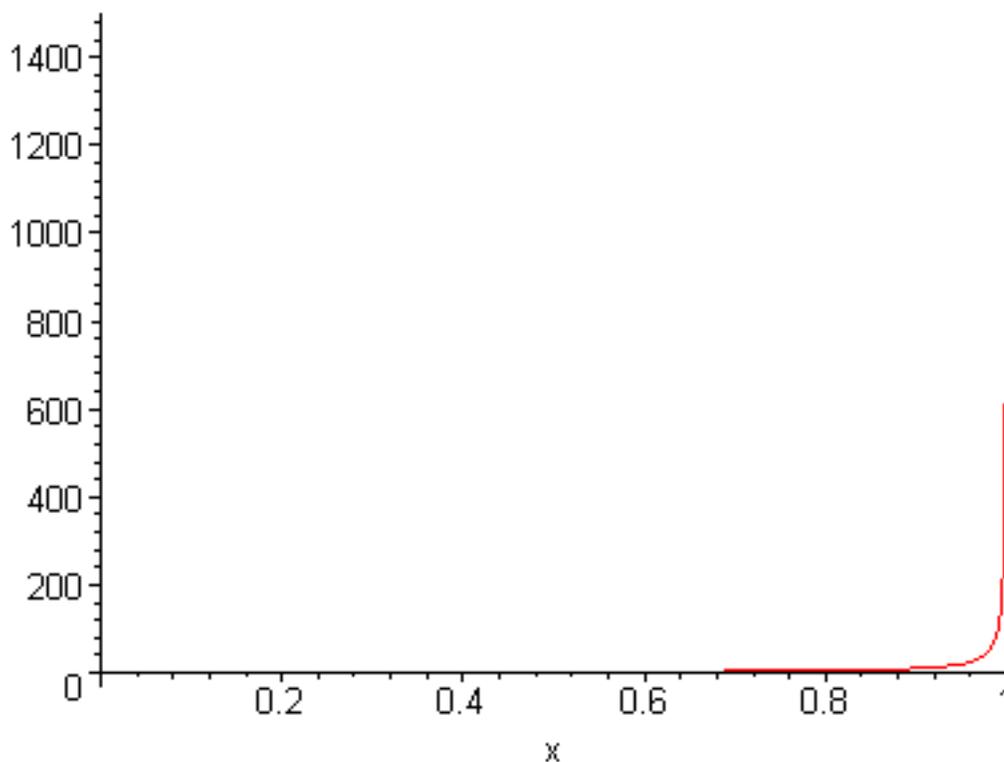
$$\frac{\partial}{\partial x} y = y^2$$

Keeping in mind that with a first power on the right we got the exponential function, this equation says that if y starts out positive, y should increase even more than it did before, i.e., get bigger faster than the exponential function. That would have to be pretty impressive, and it is: it goes to infinity before x gets to infinity:

```
> s:=dsolve({diff(y(x),x)=y(x)^2,y(0)=1},y(x));
```

$$s := y(x) = \frac{1}{-x + 1}$$

```
> plot(rhs(s),x=0..1);
```



Problem 7.12

Now think about what should happen if $y(0)$ is negative. Make some rough sketches, then check your answer by changing the initial conditions in the example above.

>

You can play this game with second-order differential equations too. Let translate the simple harmonic oscillator equation

$$\frac{\partial^2}{\partial x^2} y = -y$$

into words. We need to remember that the second derivative means the curvature of the function: a positive second derivative means that the function curves like the smiley face of someone who is always positive, while negative curvature means that it curves like a frowny face (easy to remember, no?) And if the second derivative is large in magnitude then the smile or frown is very narrow (like a piece of string suspended between its two ends from fingers held close together) and if the second derivative is small in magnitude it is like taking the same piece of string and stretching your arms apart to make a wide smile or frown. Ok, what does this equation say if $y=1$ to start? Well, the second derivative is negative, so the function y curves downward, making y smaller, which makes the frowniness smaller, but still negative, so y keeps curving downward until it crosses $y=0$. Then with y negative the differential equation says that the curvature is positive, making y start to smile and curve upward. It doesn't curve much at first because y is pretty small in magnitude, but eventually y will have a large enough negative value that $y(x)$ turns into a full-fledged smile, stops going negative, and heads back up toward $y=0$ again. When it gets there y becomes positive, the function gets frowny and turns back around toward $y=0$, etc.. So it is no surprise that the solution of this equation is an oscillation,

$$\cos(x) \quad .$$

Problem 7.13

(a) Use this qualitative analysis to guess what the solution of this equation would look like, then check your answer with Maple.

$$\frac{\partial^2}{\partial x^2} y = y \quad \text{with } y(0) = 1 \quad \text{and} \quad \left(\frac{\partial}{\partial x} y \right) (0) = 0 \quad .$$

(b) Use this qualitative analysis to guess what the solution of this equation would look like, then check your answer with Maple (you will need to use **dsolve(...type=numeric)** and **odeplot**).

$$\frac{\partial^2}{\partial x^2} y = -y^2 \quad \text{with } y(0) = 1 \quad \text{and} \quad \left(\frac{\partial}{\partial x} y \right)(0) = 0$$

(c) Same thing for this one

$$\frac{\partial^2}{\partial x^2} y = -y^3 \quad \text{with } y(0) = 1 \quad \text{and} \quad \left(\frac{\partial}{\partial x} y \right)(0) = 0$$

Problem 7.14

Think again about the baseball problem, Problem 7.9. If you look at your trajectory, and especially if you increase the initial speed of the bat up to something like 80 m/s, you will see that instead of a parabola you get a curve that starts like a parabola, but then drops almost vertically at the end. Use qualitative analysis on the x and y components of Newton's second law to explain why this occurs.

Problem 7.15

Consider again the harmonic-oscillator bound-state problem, Problem 7.10:

$$\left(\frac{\partial^2}{\partial \xi^2} \psi \right) + (\lambda - \xi^2) \psi = 0$$

Think about what its solutions qualitatively look like both for small ξ and for large ξ and show that this qualitative analysis gives roughly the right answer for what the bound state eigenfunctions look like.

[Go to top of section](#)

Nonlinear dynamics and chaos (advanced topic)

Debugging

The use of computers over the past 40 years has completely revived the stodgy old subject of classical mechanics, and the system of equations that started the whole thing are these three coupled ode's discovered by Ed Lorentz while trying to model the weather.

- > **restart;with(plots):**
- > **Eq1:=diff(x(t),t) = sigma*(y(t)-x(t));**
- > **Eq2:=diff(y(t),t) = r*x(t)-y(t)-x(t)*z(t);**
- > **Eq3:=diff(z(t),t) = x(t)*y(t)-b*z(t);**

Warning, the name changecoords has been redefined

$$Eq1 := \frac{\partial}{\partial t} x(t) = \sigma (y(t) - x(t))$$

$$Eq2 := \frac{\partial}{\partial t} y(t) = r x(t) - y(t) - x(t) z(t)$$

$$Eq3 := \frac{\partial}{\partial t} z(t) = x(t) y(t) - b z(t)$$

where the constants σ , r , and b can chosen however you like. For most choices of these parameters the solutions are not particularly interesting, but Lorentz discovered that there is a range of choices for which the solution behaves in a very strange way. If we choose

- > **sigma:=10;b:=8/3;**

$$\sigma := 10$$

$$b := \frac{8}{3}$$

and then vary r from 0 up to about 30, a transition occurs from ordinary boring behavior to something so strange that it has revolutionized many fields of science, including physics. To numerically solve this set for a specific choice of r we use

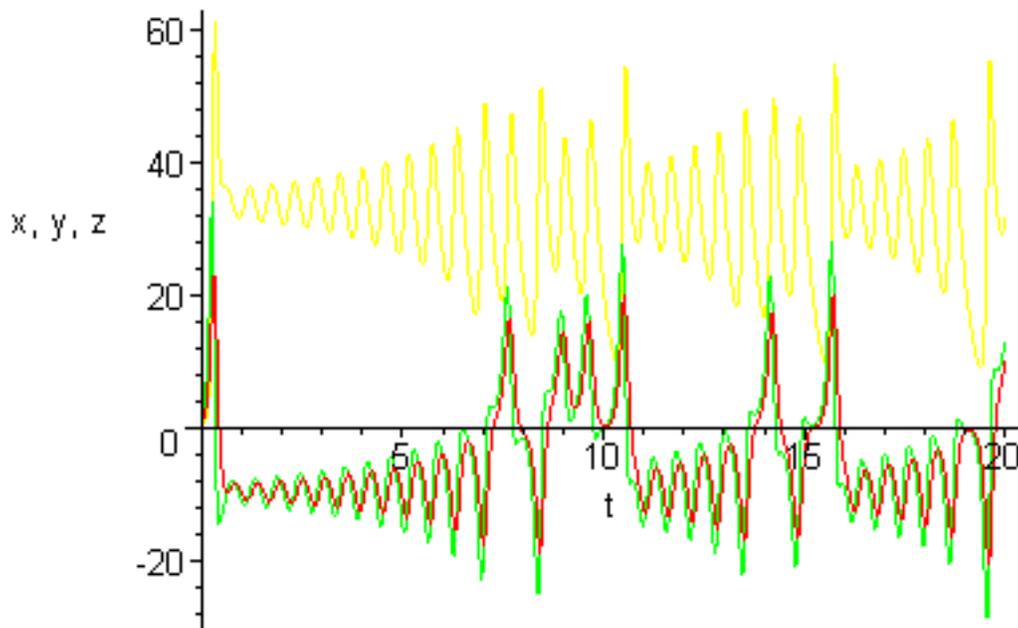
```
> r:=35;s:=dsolve({Eq1,Eq2,Eq3,x(0)=1,y(0)=1,z(0)=1},{x(t),y(t),z(t)},type=numeric);
```

and to see what the solutions look like we use **odeplot**

```
> odeplot(s,[[t,x(t)],[t,y(t)],[t,z(t)]],0..20,numpoints=200);
```

$$r := 35$$

```
s := proc(rkf45_x) ... end proc
```



Take a few minutes now and run this execution group for the following values of r : 10, 20, 28, 35, 45. For r above about 25 the oscillations form a never-repeating pattern (which you can sort of see if you run for a longer time in the **odeplot**). And what's worse, if you use slightly different initial conditions

you get a completely different pattern (this is the famous butterfly effect. To observe it change $x(0)=1$ in the initial conditions to $x(0)=1.0000001$.) This is completely unexpected. The equations that govern this behavior are completely deterministic and quite simple--how can random garbage come out of something this simple? It's truly amazing, and this kind of behavior is called *dynamical chaos* .

>

Here is another way to look at it. Instead of plotting $x(t)$, let's plot the trajectory of the solution curve $[x(t), y(t), z(t)]$ as a 3-dimensional curve using Maple's **spacecurve** plot command. To use it we need to make a loop that repeatedly calls **dsolve** to generate a sequence of 3-dimensional points along the system trajectory. This sequence of points is then passed into **spacecurve** , like this:

Put the initial values of (x,y,z) in $r0$, the starting time in $t1$, the ending time in $t2$ and the number of points along the trajectory in **numpoints** ; then calculate the time step between successive points.

> **$r0:= [1,1,1]; t1:=0; t2:=10; numpoints:=400; dt:=(t2-t1)/numpoints;$**

$$r0 := [1, 1, 1]$$

$$t1 := 0$$

$$t2 := 10$$

$$numpoints := 400$$

$$dt := \frac{1}{40}$$

Set the system parameter r

> **$r:=50;$**

$$r := 50$$

>

Step the system along with repeated calls to **dsolve** , each one having for its initial conditions the ending conditions from the previous step. This is done with a for-do-end do loop, which we will study in

Chapter 8. Note: this takes a while, so be patient.

```
> for i from 1 to numpoints do
> s:=dsolve({Eq1,Eq2,Eq3,x(0)=r0[1],y(0)=r0[2],z(0)=r0[3]},{x(t),y(t),z(t)},type=numeric):
> t0:=t1+(i-1)*dt:
> tf:=t0+dt:
> sol:=s(tf-t0):
```

Store (x,y,z) in rf[i] after each step

```
> rf[i]:=[rhs(op(2,sol)),rhs(op(3,sol)),rhs(op(4,sol))]:
```

Get the next initial condition

```
> r0:=rf[i]:
```

Bottom of the loop; go back up and do it again

```
> od:
```

Turn rf into a sequence for use with spacecurve

```
> L:=[seq(rf[i],i=1..numpoints)]:
```

Plot the trajec tory

```
> spacecurve(L,orientation=[45,60]);
```



>

Now make 100 spacecurve plots, each one with a slightly different viewing angle. Then we will animate the sequence so that the trajectory rotates on the screen, making it easy to see its 3-dimensional character

```
> for i from 1 to 100 do
```

```
> theta :=45:phi:=i*3.6:
```

>

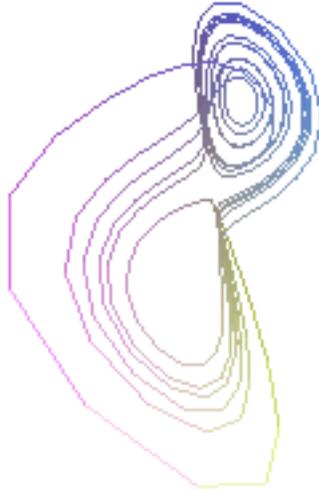
```
> p[i]:=spacecurve(L,orientation=[theta,phi]):
```

>

```
> od:
```

Plot the animation. Note: when the picture comes up and the clock goes away, click on the picture to make the animation controls appear on the toolbar.

```
> plots[display]([seq(p[i],i=1..100)],insequence=true);
```



Of course you can just click on a single space curve plot and use the cursor to rotate the picture any way you want, but I wanted you to see how to animate a spacecurve.

>

Problem 7.16

Look at how the system trajectory changes character as it changes from going to a fixed point to wandering around on the so-called strange attractor displayed by spacecurve. The transition occurs as r increases past a number around 25, or so, although if you let the system run for a long time the attractor starts to show up at lower values of r . If you try to keep numpoints too high this will take forever, and if you don't run long enough you won't be able to see where the transition occurs; use numpoints=200 and t2=20.

[Go to top of section](#)

[Go to top of chapter](#)

>

>

[Chapter8.mws](#)

- [Chapter 8: Programming -- logic, loops, and procedures](#)
 - [Logic: if-then-else-end if](#)
 - [Loops: for, do, end do, while, and break](#)
 - [Procedures: putting loops and logic to work](#)
 - [Example: plotting results from fsolve](#)
-

Chapter 8: Programming -- logic, loops, and procedures

Sometimes Maple won't have a command that does what you need to do. When this happens you have to do your own programming in Maple using logic, loops, and procedures. We will start with logic

Logic: if-then-else-end if

[Debugging](#)

The basic logic command in Maple is the **if** statement. To illustrate it, suppose that you wanted a Maple function $f(x)$ that would accept a value x and return -1 if $x < 0$, 0 if $x = 0$, and $+1$ if $x > 0$. You would write it this way:

```
> restart;
> f:=x-> if x<0 then -1 elif x=0 then 0 else 1 end if;
```

```
f:=proc(x)
option operator, arrow;
    if x < 0 then -1 elif x = 0 then 0 else 1 end if
end proc
```

Here's a test to see if it works

```
> f(-.5);f(0);f(.5);
```

0

1

(Note: this is the same as Maple's **signum** function.)

(As you will see in a minute, this **f-then-elif-else-end if** way of building functions with special cases doesn't work well in Maple. The **piecewise** syntax introduced below is the best syntax to use. I am only using this more awkward form to introduce you to logic statements.)

>

The statement defining the function *f* almost reads like English if you just translate **elif** as "else if" and **end if** as "we're done". As you can see there are three elements that go into this statement.

The first are the logical syntax commands **if ... then, elif ... then, else,** and **end if** . They have to be in this order.

The second element is the conditional statement (logical test statement). These are things like $x < 0$, etc.. Maple allows the following logical tests

< less than

<= less than or equal

> greater than

>= greater than or equal

= equal

<> not equal

When Maple encounters a test like this it evaluates it as a Boolean expression, i.e., as an expression whose value can be true or false. To help you debug things Maple has a Boolean evaluator command **evalb** . It works like this

```
> evalb(7>5);evalb(3=4);evalb(3<>4);
```

true

*false**true*

Stare at the results of executing these commands until they make sense.

Note: **and** , **or** , and **not** can also be used to combine logic statements, like this

-2<=x and x<=-1 .

There are other conjunctions as well; see **?if** and **?boolean** for more details.

The third element of the **if** statement is the set of statements that it executes. In the example above these are just the numbers -1, 0, and 1, but any valid expression could go in these positions. For instance, we could use this construct to define a function which is one thing for negative numbers and another for positive numbers, like this:

```
> g:=x-> if x<0 then 1/(1+x^2) else cos(x) end if;
```

```
g := proc(x)  
option operator, arrow;  
    if x < 0 then 1 / (1 + x^2) else cos(x) end if  
end proc
```

Test it to make sure there are no errors

```
> g(-1.5);g(2.5);
```

```
.3076923077
```

```
-0.8011436155
```

Ok, we seem to be getting numbers out. But watch what happens if you do this

```
> g(Pi);
```

```
Error, (in g) cannot evaluate boolean: Pi < 0
```

You need to be able to interpret this error message because you will be getting it occasionally. It tells you that it tried to evaluate g , but that when it did, at least one of the Boolean expressions couldn't be evaluated. This happened because Pi is not really a number in Maple, so we just asked for a comparison between the number 0 and the non-number Pi . (Pi is more of an idea than a number in Maple.) You get the same error if you try to evaluate $g(a)$, where a hasn't been given a number yet

```
> g(a);
```

```
Error, (in g) cannot evaluate boolean: a < 0
```

The problem with Pi can be fixed by remembering that Pi is tricky and always using **`evalf(Pi)`**

when you want a number:

```
> g(evalf(Pi));
```

```
-1.
```

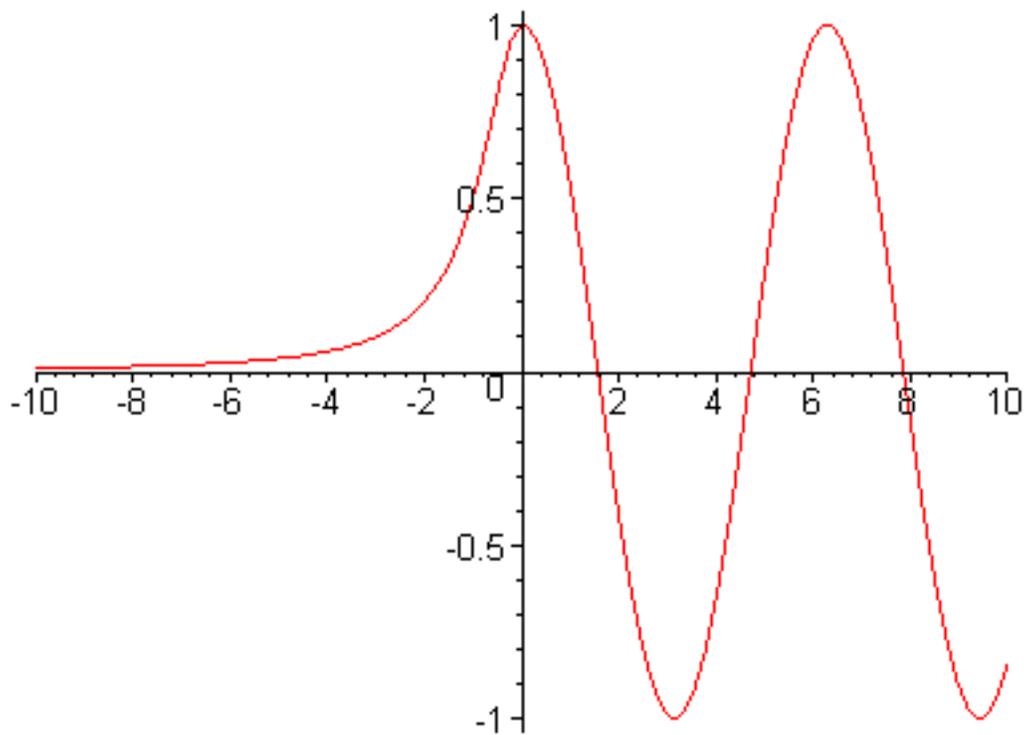
Now let's plot our function and see what it looks like

```
> plot(g(x),x=-10..10);
```

```
Error, (in g) cannot evaluate boolean: x < 0
```

Whoa, bad booleans again. Now what went wrong? Well, with the `if` statement in $g(x)$ Maple can't plot with this syntax. When x goes from the plot command into g , x is apparently not yet a number. A clue to solving this problem is found by looking at what Maple did to our statement `g:=x->...` above; it turned it into a procedure (more about these later). And Maple plots procedures using "operator notation", meaning that you don't say `g(x),x=-10..10` in the plot command, you just leave x out, like this

```
> plot(g,-10..10);
```



But if you try to integrate the function or differentiate it, you are at a dead end:

```
> int(g(x),x=-2..2);
```

```
Error, (in g) cannot evaluate boolean: x < 0
```

```
> diff(g(x),x);
```

```
Error, (in g) cannot evaluate boolean: x < 0
```

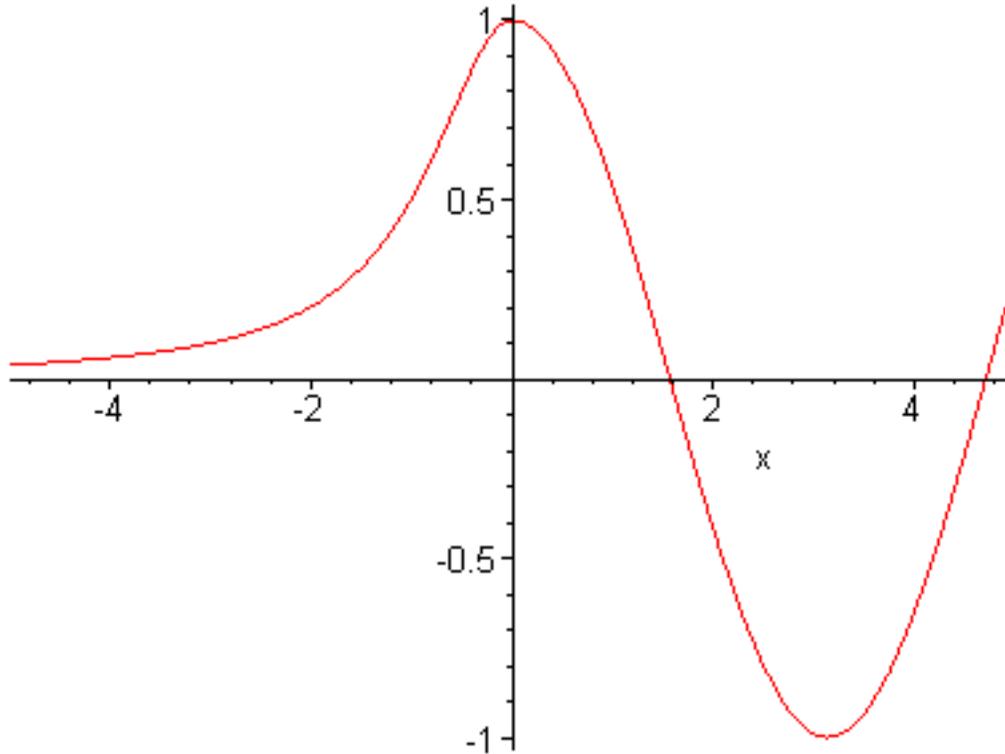
This difficulty is the reason that the **if-then-else-end if** syntax is not very convenient for defining functions in Maple. But piecewise functions like this are pretty important, so it would be good if there were some clean way to handle them. Fortunately, there is. Maple has a **piecewise** command that is used for exactly this purpose. To redefine the function $g(x)$ using **piecewise** we would do this

```
> g:=x->piecewise(x<0, 1/(1+x^2),x>=0, cos(x));
```

$$g := x \rightarrow \text{piecewise} \left(x < 0, \frac{1}{1+x^2}, 0 \leq x, \cos(x) \right)$$

Now it really is a function and you can plot it like a function

> **plot(g(x),x=-5..5);**



integrate it like a function

> **int(g(x),x=-5..5);**

$$\sin(5) + \arctan(5)$$

and differentiate it like a function

> **diff(g(x),x);**

$$\begin{cases} -2 \frac{x}{(1+x^2)^2} & x \leq 0 \\ -\sin(x) & 0 < x \end{cases}$$

So this is the best way to do piecewise functions; **if-then-elif-end if** is mostly used inside loops and procedures, which we will get to shortly.

>

Problem 8.1

Use the piecewise construct to define a Maple function which evaluates the so-called cubic B-spline function and plot it from $t=-4..4$. The function is defined this way:

$$\frac{(2+t)^3}{6} \quad \text{if } t \text{ is in the range } -2..-1$$

$$\frac{1 + (1+t)(3 + (1+t)(3 - 3(1+t)))}{6} \quad \text{if } t \text{ is in the range } -1..0$$

$B(t) =$ _____

$$\frac{1 + (1-t)(3 + (1-t)(3 - 3(1-t)))}{6} \quad \text{if } t \text{ is in the range } 0..1$$

$$-\frac{(t-2)^3}{6} \quad \text{if } t \text{ is in the range } 1..2$$

For t outside of the range $-2..2$, $B(t) = 0$.

You will discover that when you need to specify a range with a beginning and an end, like -1 to 0 , you

will need to combine two inequalities, i.e. $b < a < c$. Maple does this by using two different inequality conditions and combining them with **and**, like this:

$b < a$ and $a < c$. When you get the function defined, plot it between -4 and 4.

Problem 8.2

Integrate $B(t)$ from -3..3. Also plot its first three derivatives on the interval -3..3.

Problem 8.3

Here's an example of a problem where **piecewise** won't work cleanly and it is better to use the **if...end if** syntax.

Use this syntax to make a plot of the sine function from $-2\pi .. 2\pi$, but with the peaks and valleys all chopped off flat at height 0.9 or depth -0.9. The Maple command **signum** will be useful in doing this, as will your old friend **evalf** . Note that instead of doing logic horizontally (in x), as we do when using **piecewise** , you will be doing logic vertically (in y) instead.

[Go to top of section](#)

Loops: for, do, end do, while, and break

[Debugging](#)

Much of computing is done in loops: Maple's **sum** command has a loop at the bottom, as does numerical integration, **fsolve** , numerical differential equation solving, **dsolve(...type=numeric)** , and many more. You have seen an example of a homemade loop in the three-dimensional integration procedure in the calculus section[[Triple integrals](#)] and in the differential equation section on [Shooting](#). When Maple won't do what you want it to do, you will often find yourself writing your own loop. So that we have something concrete to talk about, let's write our own loop to sum the squares of the integers from 1 to 100. Maple's **sum** command does it this way

```
> sum(i^2,i=1..100);
```

338350

Somewhere down in Maple's depths, what is actually being done by this command is a loop. A handmade one would go like this.

Sum Loop:

First you decide where you want the answer to be stored and set that variable to zero:

```
> answer:=0;
```

```
answer := 0
```

Then you make a loop that counts from 1 to 100, and inside that loop you square the counter and add its square into answer, like this using Maple's **for..from..to..do..end do** syntax. Note that **do** means "beginning of the loop" and **end do** means "end of the loop".

```
> for i from 1 to 100 do
```

```
> answer:=answer + i^2;
```

```
> end do;
```

```
answer := 1
```

```
answer := 5
```

```
answer := 14
```

```
answer := 30
```

```
answer := 55
```

```
answer := 91
```

```
answer := 140
```

answer := 204

answer := 285

answer := 385

answer := 506

answer := 650

answer := 819

answer := 1015

answer := 1240

answer := 1496

answer := 1785

answer := 2109

answer := 2470

answer := 2870

answer := 3311

answer := 3795

answer := 4324

answer := 4900

answer := 5525

answer := 6201

answer := 6930

answer := 7714

answer := 8555

answer := 9455

answer := 10416

answer := 11440

answer := 12529

answer := 13685

answer := 14910

answer := 16206

answer := 17575

answer := 19019

answer := 20540

answer := 22140

answer := 23821

answer := 25585

answer := 27434

answer := 29370

answer := 31395

answer := 33511

answer := 35720

answer := 38024

answer := 40425

answer := 42925

answer := 45526

answer := 48230

answer := 51039

answer := 53955

answer := 56980

answer := 60116

answer := 63365

answer := 66729

answer := 70210

answer := 73810

answer := 77531

answer := 81375

answer := 85344

answer := 89440

answer := 93665

answer := 98021

answer := 102510

answer := 107134

answer := 111895

answer := 116795

answer := 121836

answer := 127020

answer := 132349

answer := 137825

answer := 143450

answer := 149226

answer := 155155

answer := 161239

answer := 167480

answer := 173880

answer := 180441

answer := 187165

answer := 194054

answer := 201110

answer := 208335

answer := 215731

answer := 223300

answer := 231044

answer := 238965

answer := 247065

answer := 255346

answer := 263810

```
answer := 272459
```

```
answer := 281295
```

```
answer := 290320
```

```
answer := 299536
```

```
answer := 308945
```

```
answer := 318549
```

```
answer := 328350
```

```
answer := 338350
```

Well, that was more information than you wanted. To fix this, end the **end do** that terminates the loop with a colon instead of a semicolon then look at answer at the end.

```
> answer:=0;
```

```
answer := 0
```

```
> for i from 1 to 100 do
```

```
> answer:=answer + i^2;
```

```
> end do:
```

```
> answer;
```

```
338350
```

And what if you want to sum the odd numbers from 1 to 99? Well, you just want to start at 1, end at 99, and step up by 2. A loop that does this using Maple's **by** syntax is

```
> answer:=0;
```

```
answer := 0
```

```
> for i from 1 by 2 to 99 do
```

```
> answer:=answer + i^2;
```

```
> end do;
```

```
> answer;
```

```
166650
```

Successive Substitution Loop:

If you know how far you want to count this is all you have to do, but sometimes you don't know how many times you want to loop. Instead, you want to loop until some condition is met. For instance, here is an interesting little trick that some of you may have noticed on your hand calculators. What happens if you just take the cosine of a number over and over and over? Let's let Maple do it 20 times and show us what it's doing as it goes along.

```
> x:=5.;
```

```
x := 5.
```

```
> for i from 1 to 20 do
```

Take the cosine of x and store the result in x so we can do it again and again

```
> x:=cos(x);
```

```
> end do;
```

```
x := .2836621855
```

```
x := .9600369303
```

```
x := .5734897327
```

$$x := .8400126809$$

$$x := .6674533830$$

$$x := .7854005360$$

$$x := .7071051035$$

$$x := .7602456870$$

$$x := .7246667298$$

$$x := .7487203837$$

$$x := .7325605057$$

$$x := .7434644380$$

$$x := .7361281031$$

$$x := .7410737901$$

$$x := .7377440894$$

$$x := .7399878116$$

$$x := .7384767772$$

$$x := .7394947924$$

$$x := .7388091199$$

$$x := .7392710309$$

It looks like the answer is converging to something. And in fact, if you think for just a minute, you will be able to see that the number it is converging to has to be a solution of the equation $\cos(x) = x$.

(Explain to your partner or to the TA why this is true.) But you can also see that 20 times was not enough to get 10 digits right; what we would really like to do is loop until successive answers are within some tolerance of each other, say $1e-8$, just to make the problem definite. Maple will let you end the loop that way too, with just a little bit of extra programming.

Choose a starting value of x and an initial fake value of $xold$, the "previous value of x "

```
> x:=5.;xold:=0;
```

```
x := 5.
```

```
xold := 0
```

Keep looping as long as $|x-xold|>1e-8$ (this is why we need the initial fake value of $xold$)

```
> for i from 1 while abs(x-xold)>1e-8 do
```

Put the new x from taking the cosine into a temporary variable $tempx$

```
> tempx:=cos(x):
```

Put the current x into $xold$ so we will be able to tell when to quit

```
> xold:=x:
```

Put the new x (called t right now) into x

```
> x:=tempx;
```

Go back up and do it again

```
> end do;
```

```
tempx := .2836621855
```

```
xold := 5.
```

$x := .2836621855$ $temp_x := .9600369303$ $xold := .2836621855$ $x := .9600369303$ $temp_x := .5734897327$ $xold := .9600369303$ $x := .5734897327$ $temp_x := .8400126809$ $xold := .5734897327$ $x := .8400126809$ $temp_x := .6674533830$ $xold := .8400126809$ $x := .6674533830$ $temp_x := .7854005360$ $xold := .6674533830$ $x := .7854005360$ $temp_x := .7071051035$

$xold := .7854005360$ $x := .7071051035$ $tempx := .7602456870$ $xold := .7071051035$ $x := .7602456870$ $tempx := .7246667298$ $xold := .7602456870$ $x := .7246667298$ $tempx := .7487203837$ $xold := .7246667298$ $x := .7487203837$ $tempx := .7325605057$ $xold := .7487203837$ $x := .7325605057$ $tempx := .7434644380$ $xold := .7325605057$ $x := .7434644380$

*temp*x := .7361281031

xold := .7434644380

x := .7361281031

*temp*x := .7410737901

xold := .7361281031

x := .7410737901

*temp*x := .7377440894

xold := .7410737901

x := .7377440894

*temp*x := .7399878116

xold := .7377440894

x := .7399878116

*temp*x := .7384767772

xold := .7399878116

x := .7384767772

*temp*x := .7394947924

xold := .7384767772

$x := .7394947924$ $tempx := .7388091199$ $xold := .7394947924$ $x := .7388091199$ $tempx := .7392710309$ $xold := .7388091199$ $x := .7392710309$ $tempx := .7389598975$ $xold := .7392710309$ $x := .7389598975$ $tempx := .7391694877$ $xold := .7389598975$ $x := .7391694877$ $tempx := .7390283084$ $xold := .7391694877$ $x := .7390283084$ $tempx := .7391234099$

$xold := .7390283084$

$x := .7391234099$

$tempx := .7390593490$

$xold := .7391234099$

$x := .7390593490$

$tempx := .7391025015$

$xold := .7390593490$

$x := .7391025015$

$tempx := .7390734336$

$xold := .7391025015$

$x := .7390734336$

$tempx := .7390930142$

$xold := .7390734336$

$x := .7390930142$

$tempx := .7390798245$

$xold := .7390930142$

$x := .7390798245$

*temp*x := .7390887092

xold := .7390798245

x := .7390887092

*temp*x := .7390827244

xold := .7390887092

x := .7390827244

*temp*x := .7390867558

xold := .7390827244

x := .7390867558

*temp*x := .7390840402

xold := .7390867558

x := .7390840402

*temp*x := .7390858695

xold := .7390840402

x := .7390858695

*temp*x := .7390846372

xold := .7390858695

$x := .7390846372$ $temp_x := .7390854673$ $xold := .7390846372$ $x := .7390854673$ $temp_x := .7390849082$ $xold := .7390854673$ $x := .7390849082$ $temp_x := .7390852848$ $xold := .7390849082$ $x := .7390852848$ $temp_x := .7390850311$ $xold := .7390852848$ $x := .7390850311$ $temp_x := .7390852020$ $xold := .7390850311$ $x := .7390852020$ $temp_x := .7390850869$

$xold := .7390852020$

$x := .7390850869$

$tempx := .7390851644$

$xold := .7390850869$

$x := .7390851644$

$tempx := .7390851122$

$xold := .7390851644$

$x := .7390851122$

$tempx := .7390851474$

$xold := .7390851122$

$x := .7390851474$

$tempx := .7390851237$

$xold := .7390851474$

$x := .7390851237$

$tempx := .7390851396$

$xold := .7390851237$

$x := .7390851396$

```
tempx := .7390851289
```

```
xold := .7390851396
```

```
x := .7390851289
```

```
tempx := .7390851361
```

```
xold := .7390851289
```

```
x := .7390851361
```

This is called successive iteration, and it is used in computing a lot. It would be a good idea for you to look at the structure of this loop carefully to see how it is built. In particular, pay attention to when each variable is computed and where it is stored. You may also notice that I tried in the loop to have only x print by using colons on the t and xold lines; this didn't work and I don't know how to fix it.

>

Well, those are the two basic kinds of loops. For additional information see **for**.

Problem 8.4

Write a loop that evaluates the power series for the cosine function, $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^{\left(\frac{n}{2}\right)} x^n}{n!}$

for n even. Give x a value outside the loop, use the **by** syntax to get even numbers, and keep looping until the next term added to the sum is less than $1e-8$. To do this you will need to know how to make a loop that adds. There is one at the beginning of this section, right here: [Adding loop](#)

Problem 8.5

Write a loop that evaluates the Fourier series $f(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \sin((2n+1)\theta)}{2n+1}$. Give θ a

value outside the loop and keep looping until the magnitude of the terms being added drop below $1e-4$ in magnitude. Since the sine function is never greater than 1 in magnitude, this means that you only need to

apply the **while** test to $\frac{1}{2n+1}$. (It is bad to test on the whole term with the sine included because

the sine might just happen to be really small and cause the loop to terminate too soon.)

Problem 8.6

Write a loop to solve the equation $x = e^{(-x)}$ by successive substitution. Get the answer to 8

significant figures. To see how to build a successive substitution loop go here (just above): [Successive substitution loop](#)

Problem 8.7 (Secant Method)

Here's a loop that computational physicists use all the time. It's called the secant method and it's a way of solving hard equations in one variable. In fact, Maple almost certainly uses a fancy variation of this technique inside the **fsolve** command. I also used it in the shooting procedure for differential equations in Chapter 7. Here's the idea.

Suppose you want to solve a hard equation of the form $f(x) = 0$, for x . The secant method starts by asking you for two reasonably close guesses x_1 and x_2 . Then you evaluate the function at each value of x to get $f_1 = f(x_1)$ and $f_2 = f(x_2)$. You learned in high school algebra what to do with two ordered pairs (x_1, f_1) and (x_2, f_2) like this: use the two-point line formula to put a

straight line through them: $y - f_2 = \frac{(f_2 - f_1)(x - x_1)}{x_2 - x_1}$. But what we have just built is a straight-

line approximation to the function $f(x)$, so to get an approximation to the value of x that solves our hard equation, we just solve the line formula above for the value of x that gives $y = 0$ and call it x_3 , our next best guess at where the solution is.

> **restart;**

> **Eqline:=0-f2=(f2-f1)/(x2-x1)*(x-x2);**

$$Eqline := -f_2 = \frac{(f_2 - f_1)(x - x_2)}{x_2 - x_1}$$

> **x3:=solve(Eqline,x);**

$$x_3 := \frac{f_2 x_1 - x_2 f_1}{f_2 - f_1}$$

>

This is a good algebraic expression, but not a good numerical one because as we get close to the root when both x_1 and x_2 , and f_1 and f_2 , are close to each other we get x_3 by almost dividing zero by zero. This can cause floating point troubles, so it is better to rearrange this formula so that it looks like this:

$$x_3 = x_2 - \frac{f_2(x_2 - x_1)}{f_2 - f_1}$$

With this form we get the next best guess by adding something small to the previous best guess and we get in less numerical trouble. It is true that both $(x_2 - x_1)$ and $(f_2 - f_1)$ are small, so we are dividing zero by zero again, but the values of f_1 and f_2 in the numerator are extra small because

we are close to $f(x) = 0$, so the sins we commit in dividing zero by zero don't affect the answer as much.

Ok, here is finally the problem: Write a loop that uses the secant method to solve the equation $\tan(x) - 2x = 0$ for a root near $x = 1$. (Make your loop similar to a [Successive substitution](#)

[loop](#).) In the loop you will find that it is desirable to jump out of the loop if $f_2 = f_1$ to as many significant digits as Maple is keeping (typically 10; see online help about **Digits**) because this will keep you from dividing by zero. This is hard to do with **while** , but fortunately there is a way to jump out using the **break** command.

Here is a stupid example: suppose you were looping from 1 to 100 but you wanted to jump out when i was 20. You would use

```
> for i from 1 to 100 do
> if i=20 then break end if;
> end do;
```

Notice that after the jump, i has the value it had when the jump occurred.

```
> i;
20
>
```

You will want to use this in the secant method procedure you are developing here. To test your use of it in your loop, keep repeating the calculation until the value of f2 is less than 1e-10 by using **while** ; this should make division by zero happen so you will need the **break** command to protect your loop.

You may discover, however, that getting Maple to divide by zero in this problem is difficult. If it won't cooperate, don't worry about it. The **break** statement is still a good idea in general, though, so I want you to practice putting it in.

[Go to top of section](#)

Procedures: putting loops and logic to work

Debugging

Well, you have probably noticed that loops and logic are messy. They take up a lot of space in your worksheet and to do them over and over you have to keep moving the cursor around. Maple has provided a way to put complicated stuff like this into something that works like a function so you can just pass values into it and get answers back out. The structure that does this is the procedure, and you have already encountered some of these in earlier sections. Now you get to learn how to write your own. Here is the basic structure.

> **restart;**

Assign the procedure to a name and define an argument list for it

> **f:=proc(x,y)**

Declare the names of local variables, those only used within the procedure

> **local a,b,z;**

Declare the names of global variables, those from outside to be used inside

> **global d;**

Now come the Maple statements that get the job done. In this case we use local a and b, global d, and passed in x and y to compute function value z, which is local. So how does Maple know what value to give us back? *The returned value is always the last thing computed in the procedure.*

> **a:=3.;b:=17.;**

> **z:=a*b*x*y*d; # last value**

> **end;**

f:=proc(x,y)

local a, b, z;

global d;

a := 3.; b := 17.; z := a*b*x*y*d

end proc

```
> d:=31.;
```

```
d := 31.
```

```
> f(1,2);
```

```
3162.
```

```
>
```

To illustrate how a procedure works, here is one that simply evaluates the cosine of an angle in degrees.

```
> restart;
```

```
> cosdeg:=proc(theta)
```

```
> local phi,answer;
```

```
> phi:=theta*Pi/180.;
```

```
> answer:=cos(phi);
```

```
> end;
```

```
cosdeg := proc(theta)
```

```
local phi, answer;
```

```
phi := .00555555555556*theta*Pi; answer := cos(phi)
```

```
end proc
```

Now test it

```
> yy:=cosdeg(45.);
```

```
yy := cos(.2500000000 Pi)
```

Because a procedure is a program that you write, it will usually fail. (I am not trying to insult you. This sentence is also true: "When scientists and engineers write procedures, they usually fail.") And when it fails you will need to see what it is doing to find out how to fix it. The Maple command to tell a

procedure to reveal to you its inner workings is the command **trace** . To debug the procedure above use

```
> trace(cosdeg);
```

```
cosdeg
```

then execute it like this

```
> cosdeg(45);
```

```
{--> enter cosdeg, args = 45
```

```
phi := .2500000000 pi
```

```
answer := cos(.2500000000 pi)
```

```
<-- exit cosdeg (now at top level) = cos(.2500000000*pi)}
```

```
cos(.2500000000 pi)
```

And then when it is working right and you don't want to see all of this stuff every time you use it type

```
> untrace(cosdeg);
```

```
cosdeg
```

```
>
```

Problem 8.8

Notice also that you didn't get what you probably expected out of this cosdeg procedure. Fix it so that it gives a floating point number back. Make sure that your fix takes place inside the procedure and not in the call you make to it.

If you look at how cosdeg is used you can see that a procedure is used just like a function. But because it is a self-contained little program it can be a really complicated function. And this is, in fact, the usual reason for writing a procedure. You have to be careful, though, because functions defined this way may

not be compatible with other Maple commands like **plot** , **int** , **sum** , **dsolve** , etc.. In addition, procedures are usually slower than the built-in Maple commands.

Problem 8.9

Write a procedure that computes the cubic B-spline function described in Problem 8.1. Plot your procedure (function) from -3..3. Note: if you are going to plot the result of a procedure the answer has to be returned as the last result, and the only arguments that can be passed in are the appropriate independent variables, i.e. for a one-dimensional function you would use **B:=proc(x)** , for 2-d use **C:=proc(x,y)** , etc.. When you plot output from a procedure, you can treat it like a function using the **plot(f(x),x=a..b)** command, or you can use Maple's operator notation with the name of x left out: **plot(B,-3..3)** .

Problem 8.10

You can also use procedures to perform a task that you want to do a whole bunch of times while you change some parameter. Write a procedure that takes as inputs the parameter Nterms and the angle θ and returns the value of the Fourier series $f(\theta)$ given in Problem 8.5. Use Maple's **sum** command inside the procedure and keep terms up to $n=Nterms$. Then make three plots of $f(\theta)$ vs. θ with $Nterms=10$, $Nterms=100$, and $Nterms=400$. This will give you an instructive way of seeing how the Fourier representation improves as you keep more terms. Because your procedure has more than one input variable, you will need to use a plot command of form **plot(f(Nterms,theta),theta=0..Pi)** .

Problem 8.11

Write a procedure that takes as input the parameter α , then solves the differential equation

$$\frac{\partial^2}{\partial x^2} f(x) = -f(x)^3 \quad \text{with } f(0) = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial x} f \right) (0) = \alpha .$$

Let the procedure return $f(1)$ as a

single number (you will likely need to use trace to get this right). Then try different values of α and

see if you can find at least 2 values of α (other than 0) for which $f(1) = 0$. Find these 2 values of α accurate to 5 significant figures.

This is a pretty hard problem, so here are some hints. You will want to put **dsolve(...type=numeric)** into the procedure and have it return a number. But the output from the numerical version of **dsolve** is a set of equations, like this

```
> restart;
```

```
> f:=dsolve({diff(y(t),t$2)=-y(t),y(0)=2,D(y)(0)=0},{y(t)},type=numeric);
```

```
f:=proc(rkf45_x) ... end proc
```

```
> g:=f(1);
```

```
g :=
[ t = 1., y(t) = 1.0806046976628,  $\frac{\partial}{\partial t} y(t) = -1.6829420734690$  ]
```

When we encountered this before we used the assign statement to get the number assigned to y(t), like this

```
> assign(g);
```

```
Error, (in assign/internal) invalid left hand side in assignment
```

This failed because the derivative symbol is not a valid variable to assign to. So now you have two choices. (i) Change the second order ode into a coupled set of ode's and use the assign statement to extract the number or (ii) learn how to use Maple's **op** and **rhs** commands to extract the number. The command **op** picks out elements of almost anything Maple generates: expressions, functions, lists, sets, etc.. Go back up to the restart above, execute the **restart**, the **dsolve**, and the **g:=f(1)**; but *do not execute the assign statement*. Then in the statement below change the 1 to 2 and then to 3, and see what you get back

```
> op(1,g);
```

```
Error, wrong number (or type) of parameters in function diff
```

The number $f(1)$ is really close now; all we have to do is get it out of the equation, and that's just what **rhs** does: it extracts the right-hand side of an equation. So see what happens when you do this

```
> rhs(op(2,g));
```

```
Error, wrong number (or type) of parameters in function diff
```

You will need to use this construction inside your procedure so that it can return a number.

```
>
```

Problem 8.12

Write a procedure that changes the $P_{nm}(x)$ Associated Legendre Function discussed in Problem 3.4 into a real function that returns a number for any valid choice of n and m , including $m=0$. This means that you have to solve the "differentiate zero times" problem discussed in Chapter 3. [Associated Legendre functions](#) Test your procedure sufficiently that you know it is working correctly. (a) First write the procedure so that it returns an expression in x , then (b) change it so it returns a floating point number.

[Go to top of section](#)

Example: plotting results from fsolve

[Debugging](#)

Here is something that it is a little technical, but I run up against it all the time, so maybe you will too.

Suppose you have a hard equation to solve that has a variable parameter k in it, like this

$$\cos(x) = kx$$

where k varies from 0 up to 20. I want to build a function $F(k)$ that returns the solution x of

this equation for any k that I give it, and then I want to plot this function $F(k)$. The only way I know to do this is to write a procedure. After the Maple commands inside the procedure below have executed, F is assigned the last result calculated in the procedure, in this case the value of x from the **fsolve**, which is just what we want F to be.) Study the procedure below, then execute the plot.

> **restart;**

Declare F to be a procedure

> **F:=proc(k)**

Declare variables that will only be used inside the procedure

> **local s,x;**

Use **fsolve** to solve the equation using the value of k passed in through **proc(k)**

> **s:=fsolve(cos(x)-k*x,x,0..2);**

Return

> **end;**

$F :=$
proc(k) local s, x; s := fsolve(cos(x) - k*x, x, 0 .. 2) end proc

Try a few values

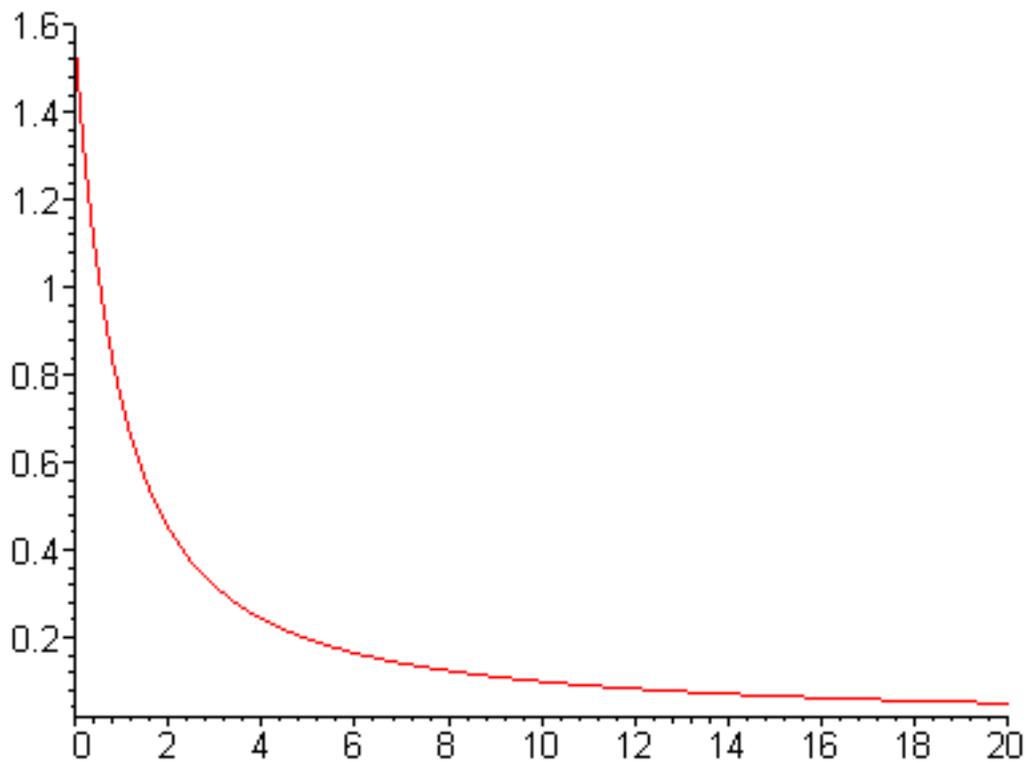
> **F(1);F(2);**

.7390851332

.4501836113

Make a plot (use operator notation--the form **plot(F(k),k=0..20)** doesn't work)

> **plot(F,0..20);**



>

Problem 8.13

Consider the polynomial $x^4 + ax^3 + x^2 + x + 1$ with a a variable parameter. You can give this polynomial to **fsolve** and it will find all four roots as long as a has a value.

> **a:=2.5;**> **P:=x^4+a*x^3+x^2+x+1;**> **s:=fsolve(P,x,complex);**

$$a := 2.5$$

$$P := x^4 + 2.5x^3 + x^2 + x + 1$$

$$s := -2.150706634, -0.7857634353,$$

$.2182350347 - .7376368563 I, .2182350347 + .7376368563 I$

If you want to select one of the roots, say the second one, you just do this

> **s[2];**

$-.7857634353$

Copy the procedure at the start of this section and modify it to plot the second root of this polynomial as a varies between 2 and 10. (Just put the Maple code in this problem into the procedure.) Note:

procedures return the last value calculated, so in your new procedure, right after the **s:=fsolve(...)** you will need the command **s[2];** to return the value of the second root.

>

Problem 8.14

(a) Write a procedure that returns the function $t(x)$ defined as the solution of the implicit equation

$$t = \frac{x e^{(-t)}}{1 + t^2}.$$

In the procedure use Maple's **fsolve** command to do the solve and when you have built it, plot $t(x)$ on the range $x=0..10$.

(b) Do the same thing as in (a), but this time use the function notation to define $t(x)$, i.e., **t:=x->fsolve(...)**. Are the two ways of doing this equivalent?

(c) Write either a function or a procedure that will give you the derivative of this implicitly defined function, i.e., get a Maple function or procedure for $\frac{\partial}{\partial x} t(x)$. Warning: do not just take the x

derivative of the equation above treating t like a constant-- t is really $t(x)$ on *both sides of the equation* . So rewrite the equation with t replaced by $t(x)$, then use **diff** to differentiate the whole

equation with respect to x . Then have Maple solve for $\frac{\partial}{\partial x} t(x)$ and evaluate the resulting

expression by modifying your procedure for $t(x)$. When you have built this procedure, use it to plot

$\frac{\partial}{\partial x} t(x)$ on the range 0..10 and verify visually that it looks like the derivative of the function you

plotted in part (a).

[Go to top of chapter](#)

>

>

[Chapter9.mws](#)

- [Algebra on matrices filled with expressions](#)

Chapter 9: Symbolic algebra

[Debugging](#)

This is what Maple was actually designed to do: symbolic mathematics. We have already seen some of its ability to do this as we have taken derivatives, integrated, and solved differential equations. I have placed this material in the last chapter because it's almost an art form and you need to be pretty comfortable with Maple before you tackle it. You can use this chapter as a source for what to try when Maple gives you answers that are correct, but not in the form you want. It may take many months to get good at this, but it is worth the effort because Maple will not make minus sign mistakes or lose factors of 2. The trick, however, is to coax it to combine precision with a sense of style and good taste. This is done via a partnership: Maple supplies the precision; you supply the style and taste. I will give you just enough of an introduction to get you started. After that it's up to you and online help.

copy and paste:

Maple almost never gives you exactly what you want, so you will still need to do some algebra by hand. But the editing capability of Maple using the mouse, copy (**ctrl-c**) and paste (**ctrl-v**) often makes it possible for you to stay in the Maple worksheet and do algebra by hand.

expand:

Here are some examples of what expand does.

> **expand((x-1)^2*(x+2));**

$$x^3 - 3x + 2$$

> **expand(cos(x+y));**

$$\cos(x) \cos(y) - \sin(x) \sin(y)$$

> **expand(exp(x-y));**

$$\frac{e^x}{e^y}$$

> **assume(x,real);expand(ln(x/(1+x^2)));**

$$-\ln(1+x^2) + \ln(x)$$

> **expand(tan(x-y));**

$$-\frac{\tan(x) + \tan(y)}{1 + \tan(x)\tan(y)}$$

This command is half of Maple's all-purpose algebra simplification duo-- **simplify(expand(%))**; . Always try this combination first when you get an algebraic expression that needs help.

combine:

This is the flip side of **expand** . The two of them together know some trig identities, but not as many as they should.

> **restart;**

> **expand(cos(x+y));**

$$\cos(x)\cos(y) - \sin(x)\sin(y)$$

> **combine(%);**

$$\cos(x+y)$$

> **expand(exp(x-y));**

$$\frac{e^x}{e^y}$$

> **combine(%);**

$$e^{(x-y)}$$

> **assume(x,real);expand(ln(x/(1+x^2)));**

$$-\ln(1+x^2) + \ln(x)$$

> **combine(%);**

$$\ln\left(\frac{x}{1+x^2}\right)$$

> **restart;**

> **expand(tan(x-y));**

$$\frac{\tan(x) - \tan(y)}{1 + \tan(x) \tan(y)}$$

> **combine(%);**

$$\frac{\tan(x) - \tan(y)}{1 + \tan(x) \tan(y)}$$

Hmm---and sometimes it doesn't work. Let's try **expand** and **simplify** and see what happens

> **expand(%);simplify(%);**

$$\frac{\tan(x)}{1 + \tan(x) \tan(y)} - \frac{\tan(y)}{1 + \tan(x) \tan(y)}$$

$$\frac{-\cos(y) \sin(y) + \cos(x) \sin(x)}{-1 + \cos(x)^2 + \cos(y)^2}$$

Well, sometimes you just can't get where you want to go.

>

simplify:

Maple tries to read your mind with this command, but doesn't always succeed. You usually have to be creative with the other commands discussed here to get exactly what you want. A good thing to try is a combination of **expand**, **simplify**, and **combine**. And don't forget that you can change the forms of expressions by using copy and paste.

> **simplify(cos(x)^2+sin(x)^2);**

1

> **simplify((x^2-2*x+1)/(x-1));**

x - 1

> **simplify(cos(x)^2-sin(x)^2);**

$2 \cos(x)^2 - 1$

> **simplify(2*sin(x)*cos(x),trig);**

$2 \cos(x) \sin(x)$

Simplify doesn't know it, but these last two can be rewritten as $\cos(2x)$ and $\sin(2x)$. **Simplify** doesn't give these to you, but for some reason **combine** does

> **combine(cos(x)^2-sin(x)^2);combine(2*sin(x)*cos(x),trig);**

$\cos(2x)$

$$\sin(2x)$$

So you will have to be creative and play around with these commands to master them.

simplify with side-relations:

You can give **simplify** some extra help by using *side-relations* (see **?simplify/siderels**). The idea is to tell Maple what identities you want to use to put your expression into the form you want. For example,

suppose you wanted to put the expression $\cos(x)^4 - 3 + \cos(x)^2$ into a new form involving $2x$

instead of x by using the identity $\cos(x)^2 = \frac{1 + \cos(2x)}{2}$. You could do this

```
> f:=cos(x)^4-3+cos(x)^2;
```

$$f := \cos(x)^4 - 3 + \cos(x)^2$$

```
> simplify(f,{cos(x)^2=(1+cos(2*x))/2});
```

$$-\frac{9}{4} + \frac{1}{4} \cos(2x)^2 + \cos(2x)$$

(Notice that the side-relations must be inside curly brackets.) You may think that this is just a substitution (see the **subs** command, discussed below), but it isn't the same:

```
> subs(cos(x)^2=(1+cos(2*x))/2,f);
```

$$\cos(x)^4 - \frac{5}{2} + \frac{1}{2} \cos(2x)$$

Using side-relations is like an intelligent substitution where Maple tries to change everything it can using the identity you gave it. The side-relations go in curly brackets, as shown in the example above, and several identities can be specified at once using the syntax **simplify(expression,{side1,side2,side3});**

```
> f:=cos(x)^4+1-cos(x)^2-sin(x)^2+3*sin(x)^6;
```

$$f := \cos(x)^4 + 1 - \cos(x)^2 - \sin(x)^2 + 3 \sin(x)^6$$

> **s1:=cos(x)^2=(1+cos(2*x))/2;**

$$s1 := \cos(x)^2 = \frac{1}{2} \cos(2x) + \frac{1}{2}$$

> **s2:=sin(x)^2=(1-cos(2*x))/2;**

$$s2 := \sin(x)^2 = \frac{1}{2} - \frac{1}{2} \cos(2x)$$

> **simplify(f,{s1,s2});**

$$\frac{5}{8} + \frac{11}{8} \cos(2x)^2 - \frac{3}{8} \cos(2x)^3 - \frac{5}{8} \cos(2x)$$

sort:

Sort puts expressions, especially polynomials, in order from high powers down to low

> **sort(2-3*x^2-x+x^4);**

$$x^4 - 3x^2 - x + 2$$

> **sort(1-cos(x)^2+cos(x));**

$$-\cos(x)^2 + \cos(x) + 1$$

factor:

Factor tries to write polynomials as products of terms

> **factor(x^2-6*x+8);**

$$(x - 2)(x - 4)$$

collect:

This is sort of a half-way factor command. Perhaps you see a common factor within a larger expression and you would like to collect together all terms that contain it, like the $\ln(y)$ term below. Then do this

```
> F:=x*ln(y)-5*x + 3*ln(y)+5*exp(y);
```

$$F := x \ln(y) - 5x + 3 \ln(y) + 5 e^y$$

```
> collect(F,ln(y));
```

$$(x + 3) \ln(y) - 5x + 5 e^y$$

And if you want to collect more than one factor at once, give **collect** a list of factors, like this

```
> F:=a*x*sin(y)+b*a*x/cos(y);
```

$$F := a x \sin(y) + \frac{b a x}{\cos(y)}$$

```
> collect(F,[x,a]);
```

$$\left(\sin(y) + \frac{b}{\cos(y)} \right) a x$$

solve:

We have seen this before; it solves an equation for the specified variable. If the variable to solve for is obvious because there is only one, you don't have to give it; and if you just give solve an expression without an = sign, it solves it by setting it equal to zero.

```
> solve(x^2-6*x+8);
```

4, 2

> **solve(ln(y/x)=3,y);solve(ln(y/x)=3,x);**

$$e^3 x$$

$$\frac{y}{e^3}$$

> **solve(1-tan(x)^2=-2,x);**

$$-\frac{1}{3}\pi, \frac{1}{3}\pi$$

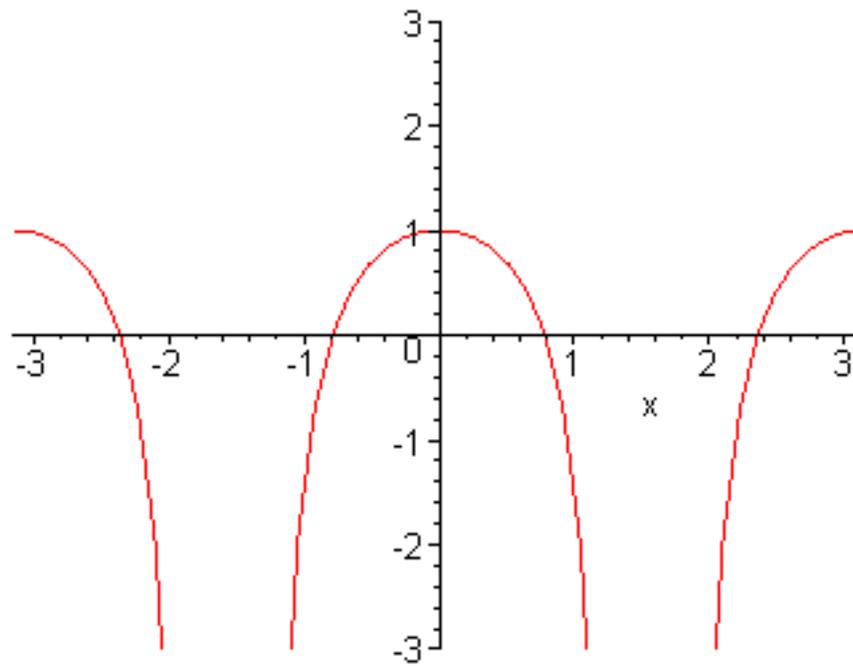
Can you spot the difference between this next one and the previous one?

> **solve(1-tan(x)^2=-2.,x);**

$$-1.047197551, 1.047197551$$

If you give Maple an equation that has no solution, it just hands it back to you, or gives you nothing.

> **plot(1-tan(x)^2,x=-Pi..Pi,-3..3);**



> **solve(1-tan(x)^2=2,x);**

As you can see from the plot the function $1 - \tan(x)^2$ can never be 2, so Maple can't solve the equation.

>

normal:

The normal form for rational functions is to have a common denominator

> **normal(1/x + 1/(x+1));**

$$\frac{2x + 1}{x(x + 1)}$$

And what happens if you try to expand this result?

> **expand(%);**

$$2 \frac{1}{x + 1} + \frac{1}{x(x + 1)}$$

It's expanded, but not back to where it came from. To get it back to where it started you have to be more specific and use the **convert** command, which is coming up.

numer and denom:

Maple will let you refer directly to the numerator or denominator of a fractional expression, like this

```
> denom((1+x)/(3-x^2));
```

$$-3 + x^2$$

```
> numer((1+x)/(3-x^2));
```

$$-x - 1$$

but with perhaps not exactly the form you expected. You actually have much more control over parts of expressions like this by using the Maple editor's copy and paste ability:

```
> (1+x)/(3-x^2);
```

$$\frac{x + 1}{3 - x^2}$$

Use the mouse to highlight the numerator of the expression above, copy it (**ctrl-c**), then paste (**ctrl-v**) in this execution group

```
>
```

op:

This command picks out pieces of expressions. Its syntax is **op(n,expr)** , which picks out the nth piece of the expression expr. Just experiment with it whenever you need to take something apart.

```
> kk:=(1+x^2)/(sin(x)+tan(x))*cosh(x)/(1+tanh(x));
```

$$kk := \frac{(1 + x^2) \cosh(x)}{(\sin(x) + \tan(x)) (1 + \tanh(x))}$$

> **op(1,kk);op(2,kk);op(3,kk);op(4,kk);**

$$\frac{1 + x^2}{\sin(x) + \tan(x)}$$

$$\cosh(x)$$

$$\frac{1}{1 + \tanh(x)}$$

>

subs:

This is the substitute-one-variable-for-another command

> **Eq1:=cos(x)^2-5*cos(x)+1;**

$$Eq1 := \cos(x)^2 - 5 \cos(x) + 1$$

> **subs(cos(x)=y,Eq1);**

$$y^2 - 5y + 1$$

> **Eq2:=exp(2*x)-t^2=x*t;**

$$Eq2 := e^{(2x)} - t^2 = xt$$

> **subs(x=xi,t=tau,Eq2);**

$$e^{(2\xi)} - \tau^2 = \xi \tau$$

As you use **subs** you may become frustrated because it seems stupid, like this:

```
> f:=x^4-3*x^2+5;
```

$$f := x^4 - 3x^2 + 5$$

```
> subs(x^2=y,f);
```

$$x^4 - 3y + 5$$

Why couldn't it recognize that $x^4 = y^2$? Well, **subs** is just very literal, not algebraic. There is,

however, an algebraic form of **subs** called **algsubs** which is much smarter. You will probably have lots better luck with it because it recognizes that your x^2 substitution was really an x substitution.

```
> algsubs(x^2=y,f);
```

$$-3y + y^2 + 5$$

```
>
```

convert:

Finally, there is the **convert** command, which is actually many commands depending on what kind of conversion you want to do. Use **?convert** to see all of the possibilities. When you get stuck and the commands discussed above don't do the job, try looking through the list of **convert** options in online help. Here are a few of what I think are useful or interesting options.

Put rational functions in partial fraction form with the **parfrac** option

```
> convert((2*x+1)/(x*(x+1)),parfrac,x);
```

$$\frac{1}{x} + \frac{1}{x+1}$$

Convert rational functions to continued fraction form with **confrac**

> **convert((2*x+1)/(x*(x+1)),confrac,x);**

$$2 \frac{1}{x + \frac{1}{2} - \frac{1}{4} \frac{1}{x + \frac{1}{2}}}$$

Express functions in exponential form with **exp**

> **convert(sin(x),exp,x);**

> **convert(sinh(x),exp,x);**

$$\frac{-1}{2} I \left(e^{(Ix)} - \frac{1}{e^{(Ix)}} \right)$$

$$\frac{1}{2} e^x - \frac{1}{2} \frac{1}{e^x}$$

Convert expressions with sines and cosines into tangent form with **tan**

> **restart;**

> **a:=convert((2+cos(x)^2)/(2-sin(x)^2),tan);**

$$a := \frac{2 + \frac{\left(1 - \tan\left(\frac{1}{2}x\right)\right)^2}{2}}{\frac{\left(1 + \tan\left(\frac{1}{2}x\right)\right)^2}{2 - \frac{4 \tan\left(\frac{1}{2}x\right)}{\left(1 + \tan\left(\frac{1}{2}x\right)\right)^2}}}$$

> **a:=simplify(a);**

$$a := \frac{1}{2} \frac{3 + 2 \tan\left(\frac{1}{2}x\right) + 3 \tan\left(\frac{1}{2}x\right)^2}{1 + \tan\left(\frac{1}{2}x\right)^4}$$

Convert expressions with tangents into sine-cosine form with **sincos**

> **convert(a,sincos);**

$$\frac{1}{2} \frac{3 + \frac{2(1 - \cos(x))^2}{\sin(x)^2} + \frac{3(1 - \cos(x))^4}{\sin(x)^4}}{1 + \frac{(1 - \cos(x))^4}{\sin(x)^4}}$$

> **simplify(%);**

$$\frac{\cos(x)^2 + 2}{\cos(x)^2 + 1}$$

Convert English units to metric ones with **convert(metric)**

> **convert(1.*acre,metric);convert(1.*MPH,metric);**

$$4046.856422 \text{ m}^2$$

$$1.609344000 \frac{\text{km}}{\text{hr}}$$

To see what English quantities can be converted to metric try

> **?metric**

Convert expressions containing **abs** , **Heaviside** , **signum** , etc., into **piecewise** functions

> **F:=abs(x);**

> **F:=convert(F,piecewise,x);**

$$F := |x|$$

$$F := \begin{cases} -x & x < 0 \\ x & 0 \leq x \end{cases}$$

For instance, **convert** will give you a nice definition of Maple's **signum** function

> **G:=signum(x);**

> **G:=convert(G,piecewise,x);**

$$G := \text{signum}(x)$$

$$G := \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & 0 < x \end{cases}$$

Convert complex expressions to **polar** form

> **convert(x+I*y,polar);**

$$\text{polar}(|x + Iy|, \text{argument}(x + Iy))$$

Now this is probably not exactly what you had in mind. Fortunately Maple's **map** command can put it in a more useful form. The command **map** tells Maple to apply the procedure in the first slot of **map(a,b)** to each of the operands of the object in the second slot, in this case the magnitude and argument expressions. The conversion procedure is **evalc**, so we would do this

> **map(evalc,%);**

$$\text{polar}(\sqrt{x^2 + y^2}, \text{arctan}(y, x))$$

>

Map is quite powerful; see **?map** for details.

ratpoly:

The **ratpoly** option converts a series into a rational function. This is often a very convenient and compact way to get an approximate expression for a complicated function. The syntax is

convert(series, ratpoly, numdeg, dendeg)

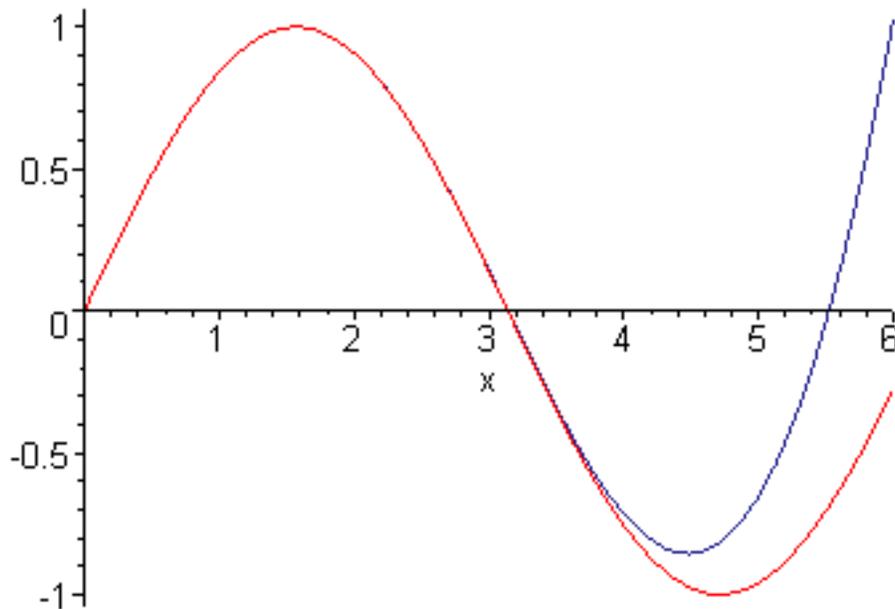
where **numdeg** is the order of the numerator polynomial and where **dendeg** is the order of the denominator polynomial. These two arguments are optional. If you don't specify them **convert** will just choose the appropriate orders for the number of terms in your series. Here are a few examples with illustrative plots using **taylor** to generate the series.

Approximate the sine function. Compare the exact result (red) with the rational approximation (blue).

```
> sinap:=convert(taylor(sin(x),x=0,10),ratpoly);
```

```
> plot([sin(x),sinap],x=0..6,color=[red,navy]);
```

$$\text{sinap} := \frac{\frac{551}{166320}x^5 - \frac{53}{396}x^3 + x}{1 + \frac{13}{396}x^2 + \frac{5}{11088}x^4}$$



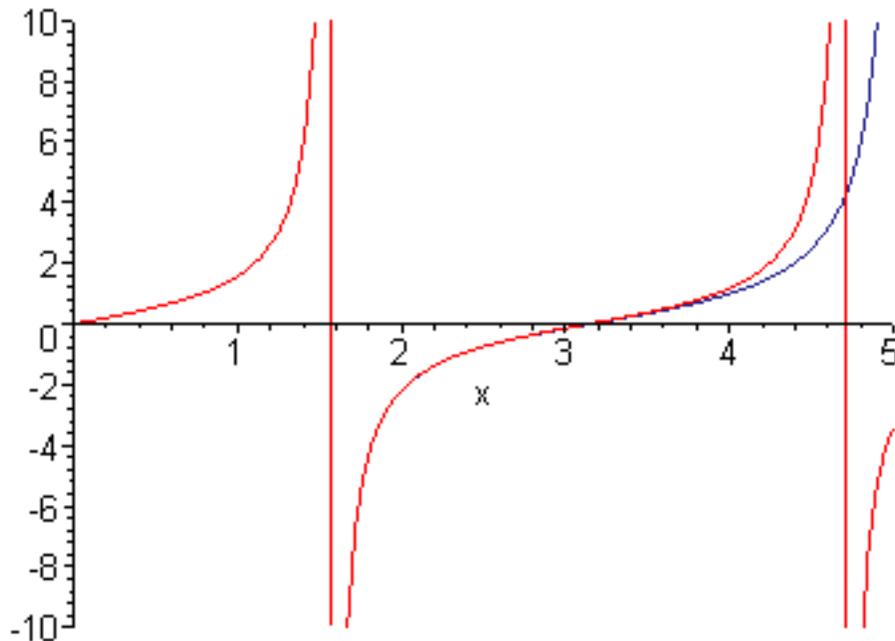
Approximate the tangent function. In this example notice how the ability of rational functions to be singular beautifully matches the singularity of $\tan(x)$ at $\frac{\pi}{2}$. The rational expression assigned below to

tanap is an excellent approximation to $\tan(x)$. (Exact in red, approximation in blue.)

```
> tanap:=convert(taylor(tan(x),x=0,10),ratpoly);
```

```
> plot([tan(x),tanap],x=0..5,-10..10,color=[red,navy]);
```

$$\text{tanap} := \frac{\frac{1}{945}x^5 - \frac{1}{9}x^3 + x}{1 - \frac{4}{9}x^2 + \frac{1}{63}x^4}$$

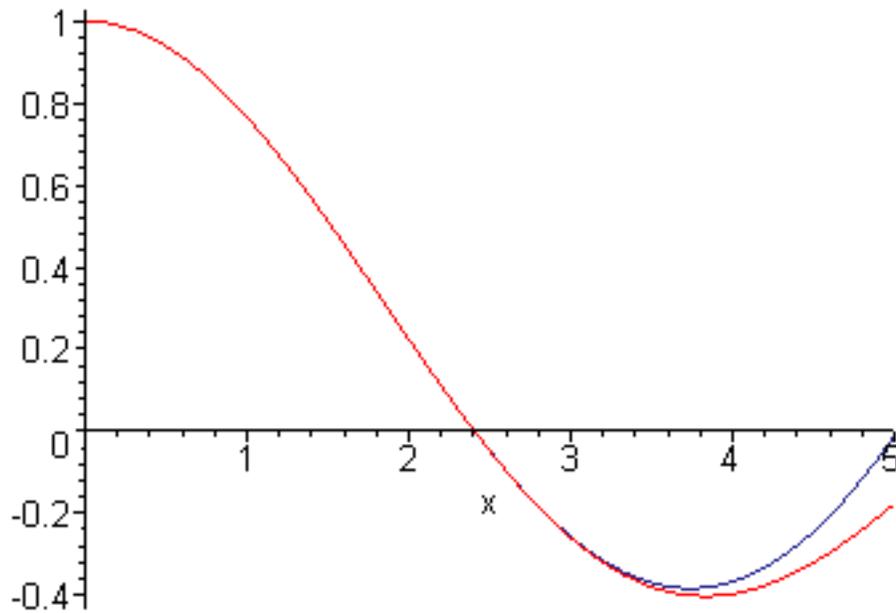


I know this may seem a little strange--why would I want to build a rational function approximation to a function I already have? Well, suppose you had to have Bessel functions in a C or Java program. You could use Maple to get a simple approximation to it using **ratpoly**, then use this approximation in your other program. (Be careful that you don't use the approximation outside of the domain where it works.)

> **J0ap:=convert(taylor(BesselJ(0,x),x=0,10),ratpoly);**

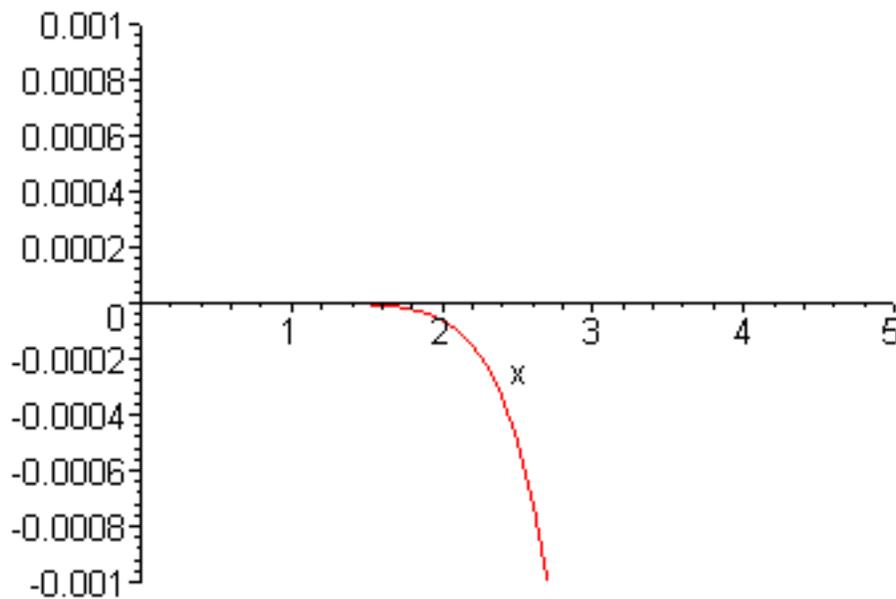
> **plot([BesselJ(0,x),J0ap],x=0..5,color=[red,navy]);**

$$J0ap := \frac{1 - \frac{17}{80}x^2 + \frac{79}{11520}x^4}{1 + \frac{3}{80}x^2 + \frac{7}{11520}x^4}$$



Or even better, plot the difference between the exact function and the approximation so you know over what range the approximation is sufficiently good.

```
> plot(BesselJ(0,x)-J0ap,x=0..5,-.001..0.001);
```



```
>
```

Strategy:

Because Maple will not do exactly what you want it to do you will get very frustrated. Perhaps the best

way to avoid ulcers as you use it is not to expect too much. In complicated problems you will need to do a lot of pencil-and-paper work to guide the calculation, but you can use Maple to do the details. Maple does lots better on small chunks of algebra than it does on big ones, so feed it bite-size pieces, then find creative ways of combining the pieces on your paper. And when you find what looks like a promising approach, use Maple to implement it. Finally, when you have used Maple, your head, and your hand to get a final beautifully simplified answer, plot it against the original ugly form to make sure that they are really the same.

Sometimes you will need to do algebra on matrices, and you will discover that it is a little tricky. Here is a section on how to handle problems like this.

Algebra on matrices filled with expressions

Maple will let you do all of the things discussed in the section above to expressions contained inside matrices and arrays, but you have to work a little harder. Suppose we have the following matrix of expressions

> **restart;with(LinearAlgebra):**

> **F:=Matrix([[x^2+x*y^2,1/x+y/x],[(x-y)*(x+y),x^2+2*x*y+y^2]]);**

$$F := \begin{bmatrix} x^2 + xy^2 & \frac{1}{x} + \frac{y}{x} \\ (x-y)(x+y) & x^2 + 2xy + y^2 \end{bmatrix}$$

and suppose we want to change its form using the algebra commands discussed in the previous section. We might try to substitute q for x in each term of the matrix using **subs**

> **subs(x=q,F);**

$$\begin{bmatrix} q^2 + qy^2 & \frac{1}{q} + \frac{y}{q} \\ (q-y)(q+y) & q^2 + 2qy + y^2 \end{bmatrix}$$

which is exactly what we wanted. (Note however: if you had used **with(linalg)** instead of **with(LinearAlgebra)** this wouldn't have worked. In general, use **with(LinearAlgebra)** whenever you can.)

Now let's try expanding each term using **expand**

> **expand(F);**

$$\begin{bmatrix} x^2 + xy^2 & \frac{1}{x} + \frac{y}{x} \\ (x-y)(x+y) & x^2 + 2xy + y^2 \end{bmatrix}$$

Why does **subs** work and not **expand** ? I don't know, but for algebra commands like **expand** that fail like this, the **map** command will do the job. This would be a good time to look up **map** , but here is basically how it works. The command **map(command,symbol)** applies the first argument (**command**) to each part of the expression(s) in the second argument (each term in the matrix in our case).

> **map(expand,F);**

$$\begin{bmatrix} x^2 + xy^2 & \frac{1}{x} + \frac{y}{x} \\ x^2 - y^2 & x^2 + 2xy + y^2 \end{bmatrix}$$

> **map(factor,evalm(F));**

$$\begin{bmatrix} x(x+y^2) & \frac{1+y}{x} \\ -(-x+y)(x+y) & (x+y)^2 \end{bmatrix}$$

etc.. With the help of map you should be able to do about anything you want to matrices and vectors containing expressions.

>

[Go to top of section](#)

Well, this is a very brief introduction to a very large subject. You will only get good at algebra in Maple by doing your homework with it in spite of the pain and frustration. Here is an example of how it's done, followed by some problems for you to practice on.

Worked example:

Here's an example from wave physics. Carefully step through it watching what happens, then proceed to the problems below. The dispersion relation for electromagnetic waves in the ionosphere is

$$\omega^2 = \omega_p^2 + k^2 c^2$$

I want to find both the phase velocity $v_p = \frac{\omega}{k}$ and the group velocity $v_g = \frac{\partial \omega}{\partial k}$ as functions of

ω . So first we solve for ω :

> **restart:Eq:=omega^2=omega[p]^2 + k^2*c^2;**

$$Eq := \omega^2 = \omega_p^2 + k^2 c^2$$

> **solve(Eq,omega);**

Hmm, that didn't work well at all. I'm going to show you what went wrong now, and it's **very important**. I was seduced by Maple's beautiful typesetting capability to write the plasma frequency ω_p with a subscript, the way it would be done in a textbook. **Do not ever use this for anything but comments in your worksheet.** When Maple got to `omega[p]` in the solve it got very confused because it has been asked to find the `p`th element of the array `omega`, but `omega` isn't an array and `p` isn't an integer. When I code I always use non-subscripted, but suggestive, variables like `wp`:

> **restart:Eq:=omega^2=wp^2 + k^2*c^2;**

$$Eq := \omega^2 = wp^2 + k^2 c^2$$

> **s:=solve(Eq,omega);**

$$s := \sqrt{wp^2 + k^2 c^2}, -\sqrt{wp^2 + k^2 c^2}$$

I just want the first one and will assign to the name w

> **w:=s[1];**

$$w := \sqrt{\omega p^2 + k^2 c^2}$$

Ok, getting the phase velocity is now not too hard:

> **vp:=w/k;**

$$vp := \frac{\sqrt{\omega p^2 + k^2 c^2}}{k}$$

But I want it as a function of ω , so now I need $k(\omega)$. No problem

> **s:=solve(Eq,k);**

$$s := \frac{\sqrt{\omega^2 - \omega p^2}}{c}, -\frac{\sqrt{\omega^2 - \omega p^2}}{c}$$

Choose the positive one and assign it to k

> **k:=s[1];**

$$k := \frac{\sqrt{\omega^2 - \omega p^2}}{c}$$

Now look at the expression for vp with this form for k

> **vp;**

$$\frac{\sqrt{\omega^2 c}}{\sqrt{\omega^2 - \omega_p^2}}$$

Maple can't simplify the square root in the top unless it knows something about ω , in this case that it is positive

```
> assume(omega>0);vp:=simplify(vp);
```

$$vp := \frac{\omega \sim c}{\sqrt{\omega \sim^2 - \omega_p^2}}$$

Now it's time to find the group velocity as a function of ω .

```
> vg:=diff(w,k);
```

Error, wrong number (or type) of parameters in function diff

Now what went wrong? Well, we did something bad that you will be tempted to do often, but which you should not do. At the beginning we had a function $\omega(k)$, but now we have solved for k and assigned a formula to it, so $\omega(k)$ is now ruined:

```
> w;
```

$$\sqrt{\omega \sim^2}$$

What we should have done is assigned the solution for $k(\omega)$ to another variable, say kk . So let's do everything all over again

```
> restart:Eq:=omega^2=wp^2 + k^2*c^2;
```

$$Eq := \omega^2 = wp^2 + k^2 c^2$$

> **s:=solve(Eq,omega);**

$$s := \sqrt{wp^2 + k^2 c^2}, -\sqrt{wp^2 + k^2 c^2}$$

I just want the first one and will assign to the name w

> **w:=s[1];**

$$w := \sqrt{wp^2 + k^2 c^2}$$

Ok, getting the phase velocity is not too hard:

> **vp:=w/k;**

$$vp := \frac{\sqrt{wp^2 + k^2 c^2}}{k}$$

But I want it as a function of ω ; no problem, just solve for $k(\omega)$

> **s:=solve(Eq,k);**

$$s := \frac{\sqrt{\omega^2 - wp^2}}{c}, -\frac{\sqrt{\omega^2 - wp^2}}{c}$$

Choose the positive one and assign it to kk

kk:=s[1];

$$kk := \frac{\sqrt{\omega^2 - wp^2}}{c}$$

Now look at the expression for v_p with this form for k by substituting kk for k

> **`vp:=subs(k=kk,vp);`**

$$vp := \frac{\sqrt{\omega^2 c}}{\sqrt{\omega^2 - \omega_p^2}}$$

Maple can't simplify the square root in the top unless it knows something about ω , in this case that it is positive

>

> **`assume(omega>0);vp:=simplify(vp);`**

$$vp := \frac{\omega \sim c}{\sqrt{\omega^2 - \omega_p^2}}$$

Now find the group velocity

> **`vg:=diff(w,k);`**

$$vg := \frac{k c^2}{\sqrt{\omega_p^2 + k^2 c^2}}$$

Substitute for k again

> **`vg:=subs(k=kk,vg);`**

$$vg := \frac{\sqrt{\omega^2 - \omega_p^2} c}{\sqrt{\omega^2}}$$

> **vg:=simplify(vg);**

$$vg := \frac{\sqrt{\omega^2 - \omega_p^2} c}{\omega}$$

>

and we are done.

Problem 9.1

(a) Express the function $\cos(10\theta)$ as a polynomial in $\cos(\theta)$

(b) Express the function $\cos(\theta)^3$ as a sum of terms containing $\cos(\theta)$ and $\cos(3\theta)$.

(c) Coax Maple into giving you the half angle identities for $\cos\left(\frac{\theta}{2}\right)$, $\sin\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$.

Hint: a single simple command (one of the ones discussed above) will do this problem for you if you give it the *squares* of sine and cosine to work with. But it isn't obvious (to me anyway) which one to try. So just experiment until you find the magic one.

Problem 9.2

The formula for the effective resistance of a certain combination of resistors is given by

$$\frac{1}{R} = \frac{1}{R_1 + \frac{1}{\frac{1}{R_2} + \frac{1}{R_3}}} + \frac{1}{R_4} + \frac{1}{R_5}$$

Beat on this with Maple until you can get a semi-nice formula for R with $R^4 R^5$ in the numerator and a sum of terms in the denominator, one of which is a fraction with denominator $R^1 R^2 + R^1 R^3 + R^2 R^3$ and with a factor of $R^4 R^5$ in the numerator, times another factor, and the other two added terms in the denominator being $R^4 + R^5$. This is an exercise in the creative use of copy and paste.

Problem 9.3

In your first physics course you studied collisions between two particles. Let's do a simple case in which the two particles are constrained to move along the x-axis. The two particles have masses m_1 and m_2 and initial velocities v_{1i} and v_{2i} with the fast particle behind the slow one so they can collide. Find the final velocities v_{1f} and v_{2f} .

(a) Elastic collisions:

The conditions that allow this problem to be solved are conservation of momentum and energy:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad \text{and}$$

$$m_1 v_{1i}^2 + m_2 v_{2i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2, \quad \text{where the } \frac{1}{2} \text{ has been cancelled out of the}$$

kinetic energy equation. Coax Maple into solving these two equations for the final velocities. You will get two solutions; interpret each set physically. Then specialize to the cases (i) $m_1 = m_2$, (ii) m_2 much larger than m_1 and $v_{2i} < v_{1i}$ so that the light particle comes up from behind and collides with the massive particle, and (iii) m_1 much larger than m_2 and $v_{2i} < v_{1i}$ so that the massive particle comes up from behind and whacks the light particle. Does each case make physical sense? Talk about it with someone until you understand each case intuitively.

Problem 9.4

In Freshman physics you also studied projectile motion with gravity obtaining the following equations describing the horizontal and vertical motion:

$$x(t) = x_0 + v_{x0} t \quad \text{and} \quad y(t) = y_0 + v_{y0} t - \frac{gt^2}{2} .$$

Suppose that x_0 and y_0 are given as

well as the initial speed v_0 and the angle θ that the initial velocity makes with the horizontal plane.

Also suppose that the elevation of the region in which the projectile will fall y_f is known. Find a

formula for the horizontal range x_f and for the "hang-time" of the projectile (the time it spends in the air). Do not be satisfied with some mess involving RootOf--get workable formulas. And if you get two solutions, figure out which one is the physically relevant one. When you get final expressions for the

horizontal range and the hang-time, choose $x_0 = 0$, $y_0 = 0$, $g = 9.8 \text{ m/s}^2$, and

$v_0 = 50 \text{ m/s}$ and make plots of hang time and range as functions of θ if the elevation of the target

region is $y_f = -200 \text{ m}$ (shooting down into a valley). Use θ in the range $-\frac{\pi}{2} \cdot \frac{\pi}{2}$ (straight

down to straight up). Carefully examine the graphs to be sure they make physical sense. In particular, why doesn't 45 degrees give the best range in this case?

Problem 9.5

The formula relating the focal length f , the object distance o , and the image distance i for a

focusing lens or mirror is $\frac{1}{i} + \frac{1}{o} = \frac{1}{f}$. Given f and o , this equation can be solved for the

distance to the image. Assume that $f = 1 \text{ m}$ (about right for the curved mirror in the ESC lobby) and

make a plot of i for o ranging from 15 m to 0.5m, about the range that is possible in the lobby of

the ESC. When i becomes negative it means that the image is virtual (meaning that it's as if there were an image behind the mirror), and because your eye has a lens in it you can see virtual images, and they appear to be at the place given by this formula. Look at your graph, then go down to the lobby and see if what you see is what you got. A good movable object to use to define o is yourself. In particular, think about what you should see if you let your face be the object and place it so that the object and the image are at the same place.

What you see in the lobby will not correspond in an obvious way with your graph because your eye has a lens in it too. This means, among other things, that you can see images at infinity! If you have a partner to argue with, this lobby exercise could be both fun and instructive.

Problem 9.6

Here is a thermodynamics problem that is a pain to do by hand, but Maple takes it right in stride.

Suppose you have a quantity of water of mass M_w with specific heat $c_w = 4185$ J/kg/K and a quantity of ice of mass M_i and specific heat $c_i = 2000$ J/kg/K. The specific heat of transformation is $L = 333000$ J/kg. If the water starts out at temperature T_w and the ice starts out at temperature T_i , and if the freezing point is at temperature $T_0 = 0$ (we are in degrees celsius) do the following.

(a) Assuming that all of the ice melts (i) find a formula for the final temperature T_f of the combined mass of water and (ii) also find a formula for the mass of ice such that the final water temperature is T_0 (the final mixture has just barely melted). In case you have forgotten, or never studied, heat, here's some help. The heat lost by the water is given by the formula

$$Q_w = M_w c_w (T_w - T_f) \quad \text{and the heat gained by the ice that melts is given by the formula}$$

$$Q_i = M_i c_i (T_0 - T_i) + M_i L + M_i c_w (T_f - T_0)$$

Just use conservation of energy and the rest is not too hard. To get definite numbers out, use $M_w = 1$ kg. (Notice that part (ii) is quite practical: given water at 20 C and ice at -10 C, how much ice does it take to bring 1 kg of water to be freezing cold?)

(b) Assuming that all of the water freezes completely and then cools down below T_0 (i) find a formula for the final temperature of the combined mass of ice and (ii) also find a formula for the mass of ice such that the final ice temperature is T_0 (the final mixture is just barely frozen). Similar formulas to those used in part (a) also apply here, except this time it is the ice formula that is simple (it just warms up as ice from T_i to T_f , while the water cools to T_0 , then freezes into ice, then cools down further as ice to T_f . Another practical question: How much ice would you need at -10 C to just barely freeze 1 kg of water at 20 C? The answer is impressive and shows that a lot of heat has to be removed to freeze water.

Problem 9.7

Light is incident from transparent region a into transparent region b, which in turn transmits light into region c, as shown below. The indices of refraction in each region are denoted by n_a , n_b , and n_c .

```
> with(plots):with(plottools):
```

```
Warning, the name changecoords has been redefined
```

```
Warning, the name arrow has been redefined
```

```
> l2:=listplot([[0,-2],[0,2]]):
```

```
> l3:=listplot([[1,-2],[1,2]]):
```

```
> t1:=textplot([-0.5,5,"a"]):
```

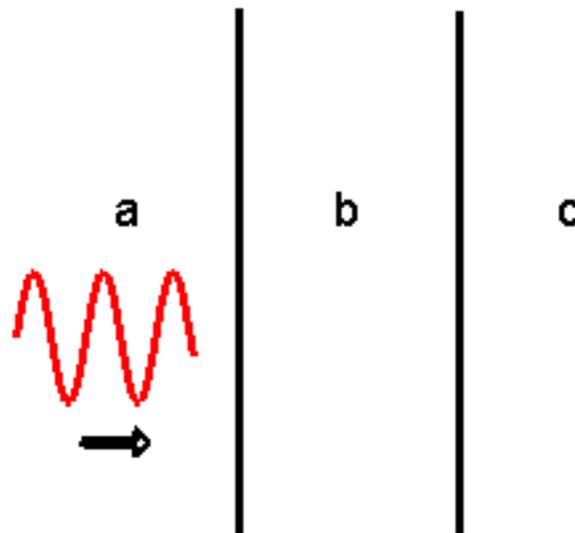
```
> t2:=textplot([0.5,5,"b"]):
```

```
> t3:=textplot([1.5,5,"c"]):
```

```
> l4:=plot(.5*sin(20*(x+1))-0.5,x=-1..-0.2):
```

```
> a1:=arrow([-1.7,-1.3],[-.4,-1.3],.05,.2,.2):
```

```
> plots[display]([t1,l2,t2,l3,t3,l4,a1],axes=none,thickness=3,view=[-2..2,-2..3],font=[HELVETICA,14]);
```



```
>
```

Maxwell's equations tell us that for normal incidence, which we assume we have here, the ratios between the incident wave in region 1 and the reflected wave in region and the transmitted wave in

region 2 are given by $\frac{E_r}{E_1} = \frac{n_1 - n_2}{n_1 + n_2}$ and $\frac{E_t}{E_1} = \frac{2n_1}{n_1 + n_2}$. For convenience we will call these

ratios R_{12} and T_{12} , respectively. Your job is to follow all of the successive transmissions and reflections in the figure above and arrive at the final ratio of the light reflected back into region a as a function of the three indices of refraction, the thickness h of region b, and the wavenumber k_b of the light in region b. To do this follow the steps outlined below.

(a) Note that the first reflection and transmission ratios are simply R_{ab} and T_{ab} . (Don't do anything except make sure you understand this statement because there are a lot more just like it coming up.)

(b) Now think about what happens to the light transmitted into region b. It first reflects from region c, is

$$(2ik_b h)$$

phase shifted by the complex phase factor $e^{(2ik_b h)}$ due to the round trip through b from a to c

and back, then is transmitted back into a. Show that the factor for this whole process, including the first

transmission, is $T_{ab} e^{(2ik_b h)} R_{bc} T_{ba}$. (This is not too difficult, but be patient--hard stuff is coming up.)

(c) Now show that each successive multiple reflection ratio is given by

$T_{ab} R_{bc} T_{ba} e^{(2ik_b h)} \left(e^{(2ik_b h)} R_{ba} R_{bc} \right)^m$ where the first multiple reflection in part (b)

corresponds to $m = 0$.

(d) Now combine all of this to get the total reflection ratio formula

$R = R_{ab} + T_{ab} R_{bc} T_{ba} e^{(2ik_b h)} \left(\sum_{m=0}^{\infty} \left(e^{(2ik_b h)} R_{ba} R_{bc} \right)^m \right)$. (Here's where the

Maple actually starts--copy this into your exercise worksheet and and go to work.)

(e) Use the formulas for the reflection and transmission coefficients and Maple's result for the sum to build a formula for $R(n_a, n_b, n_c, 2k_b h)$. Once you have it, manipulate it into a form where you can easily obtain separate equations for its real and imaginary parts. The reflected wave will be exactly zero if both real and imaginary parts vanish. You will save yourself some trouble if you define a variable $x = 2k_b h$ and work with it. It will also simplify the look of some of the equations if you let

$z = e^{(2ik_b h)}$. The formulas will become horrible, and then at the end they become simpler and

you will discover that $\sin(x) = 0$ so that $k = \frac{\lambda}{4}$, and once you know this it becomes not too hard

to get $n_b = \sqrt{n_a n_c}$, the same result you get if you ignore the problem of multiple reflections and transmissions. This is a hard problem--it took about all the Maple skill I have, and I learned some new things along the way.

[Go to top of chapter](#)

>

>

[Debugging.mws](#)

- [Debugging](#)
-

Debugging

Common problems encountered by students and their cures are collected here. You probably came here from a chapter in the text looking for help and will want to go back when you are finished. Here are hyperlinks to take you back to the start of each chapter.

[Introduction](#)

[Chapter 1: Getting started](#)

[Chapter 2: Plotting](#)

[Chapter 3: Calculus](#)

[Chapter 4: Complex numbers](#)

[Chapter 5: Linear Algebra](#)

[Chapter 6: Solving equations](#)

[Chapter 7: Differential equations](#)

[Chapter 8: Programming](#)

[Chapter 9: Symbolic algebra](#)

A. Online help

If things go wrong the first thing you should try is online help. This will not always solve your problem, but it a good place to start.

B. Unexpected results

1. If your problem is that you got an answer, but it isn't what you expected, quite often the trouble is that Maple thinks a variable can be anything, but you were thinking that it was a real number. The fix for this is

`assume(x,real,y,real,...)`

so that Maple will know the same things about the variables that you know. **However:** watch out for this problem: once you have used **assume** on a variable, however, you may have trouble having Maple give you numbers. The trouble happens when you assume something about a, use a in an expression which is assigned to another variable (say b), then give a value to a, and finally try to evaluate b. Observe:

> **restart;**

> **assume(a,real);**

> **b:=cos(a);**

$b := \cos(a)$

> **a:=2.;**

$a := 2.$

> **evalf(b);**

$\cos(a)$

2.

Now watch what happens to the same without the **assume**

> **restart;**

> **b:=cos(a);**

$b := \cos(a)$

> **a:=2.;**

$a := 2.$

> **evalf(b);**

-0.4161468365

>

I suppose that the lesson here is only to use **assume** when absolutely necessary and to be aware that this problem exists.

2. Another common problem is with **=** vs **:=**. If you use **=** when you really mean assign (**:=**) you will find that the variable to which you thought you had assigned something doesn't have anything in it.

3. Maple says it has been given an illegal character, but you don't see anything wrong with the line. (a) Look for what looks like a leading space in the command, i.e., for a space between **>** and the first character of the command. Delete this space. (b) If the problem persists, delete the entire line and retype it.

4. You get the message **Error, ';' unexpected** . Check your parentheses, they might not be balanced.

Here is a list of other common errors by category and how to fix or work around them. When you find things that drive you crazy in Maple, please email me and I will put them in this section for the benefit of other students. (ross_spencer@byu.edu)

C. Plotting Errors

empty plot

Syntax mistake when plotting an expression. For instance if we have

```
> restart;
```

```
> g:=sin(x);
```

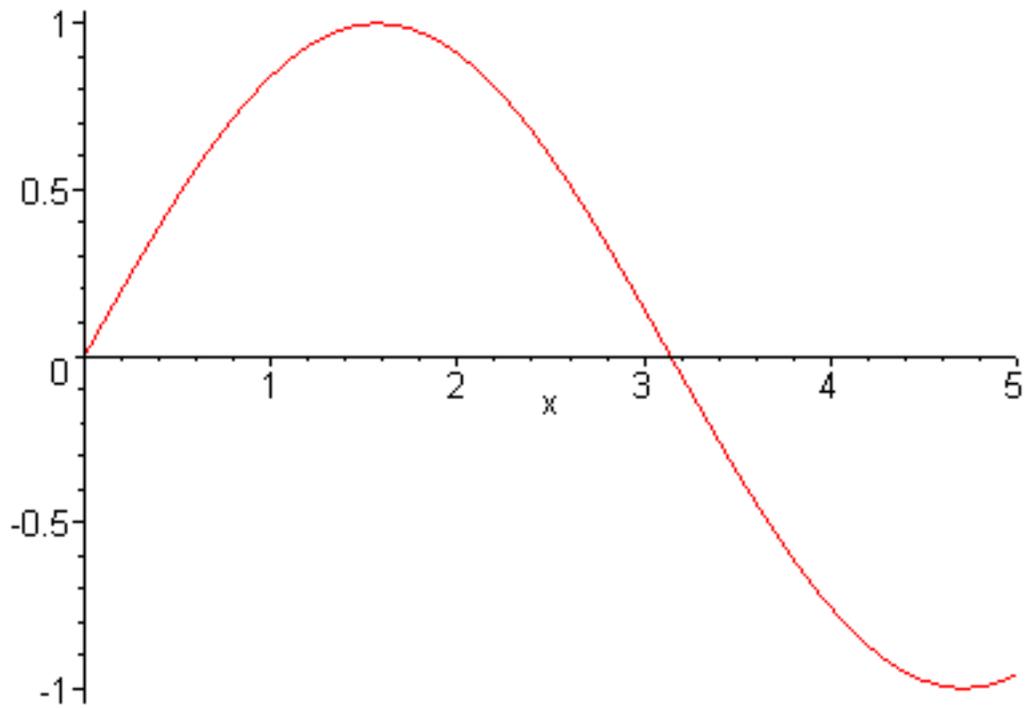
```
g := sin(x)
```

```
> plot(g,0..5);
```

Plotting error, empty plot

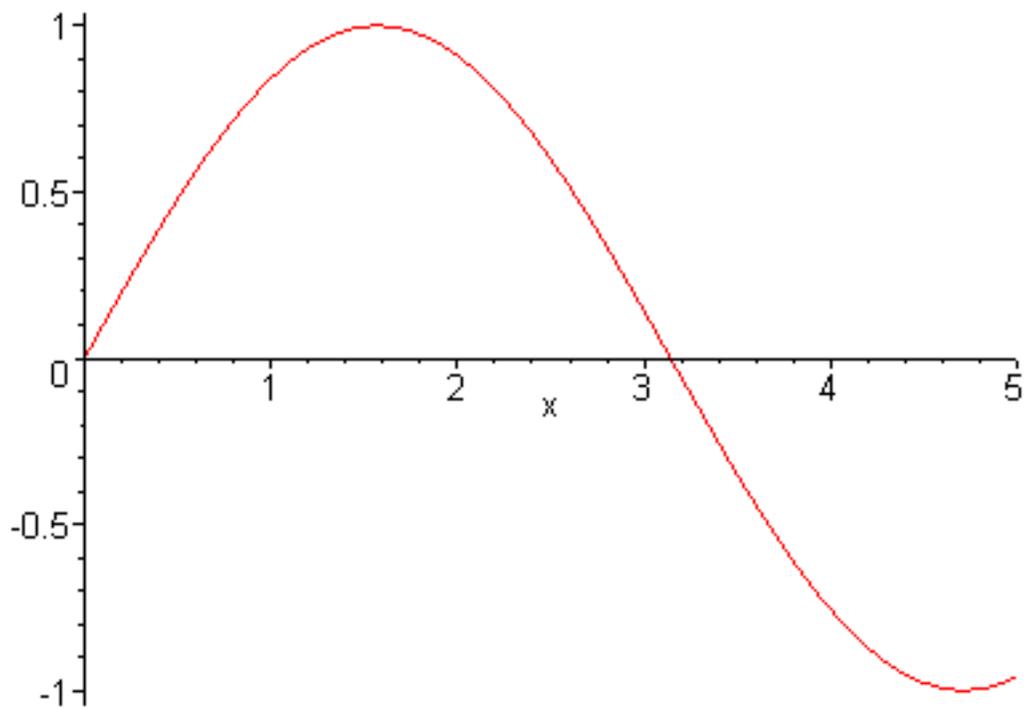
we get the empty plot error. The correct syntax for plotting an expression is

```
> plot(g,x=0..5);
```



or even

```
> plot(g(x),x=0..5);
```



axes appear, but no function is plotted

This happens when an expression, function, or procedure is to be plotted, but a parameter that needs to be assigned a value doesn't have one. For instance if we have a function f , an expression g , and a procedure p like this

```
> restart;
```

```
> f:=x->sin(x*t);
```

$$f := x \rightarrow \sin(x t)$$

```
> g:=sin(x*t);
```

$$g := \sin(x t)$$

```
> p:=proc(x)
```

```
> global t;
```

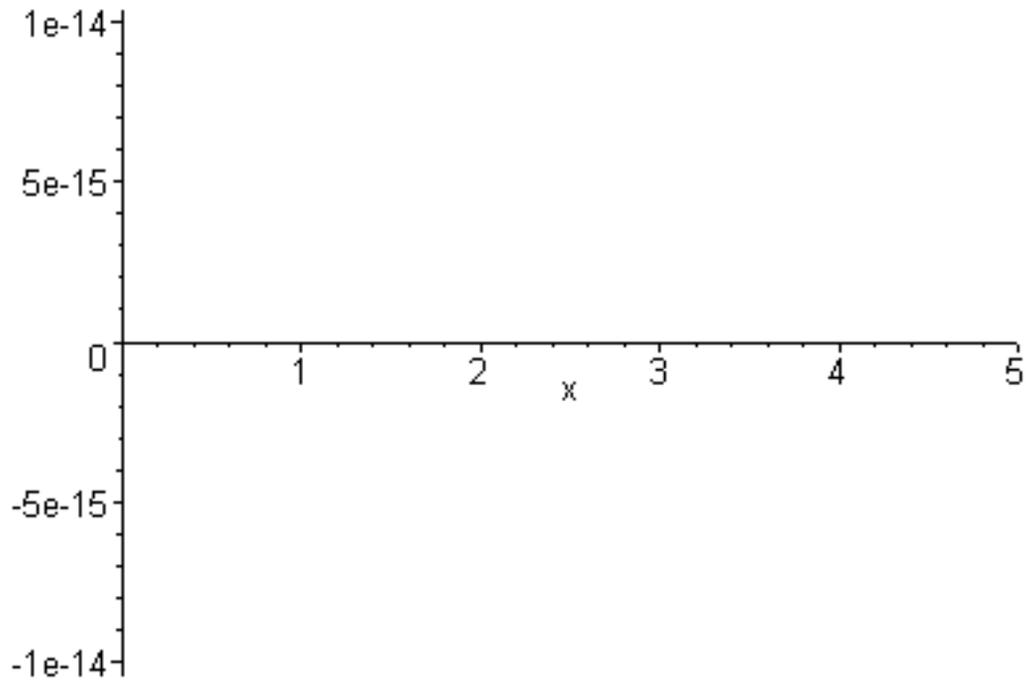
```
> sin(x*t);
```

```
> end;
```

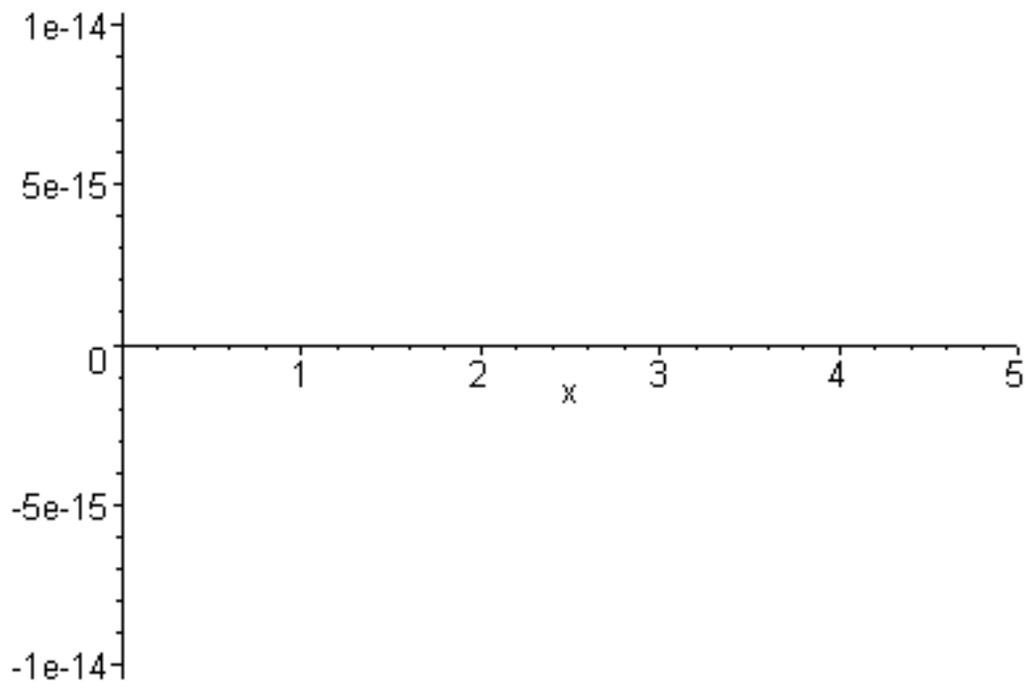
$$p := \text{proc}(x) \text{ global } t; \sin(x*t) \text{ end proc}$$

then all three of these plot commands give a zero plot

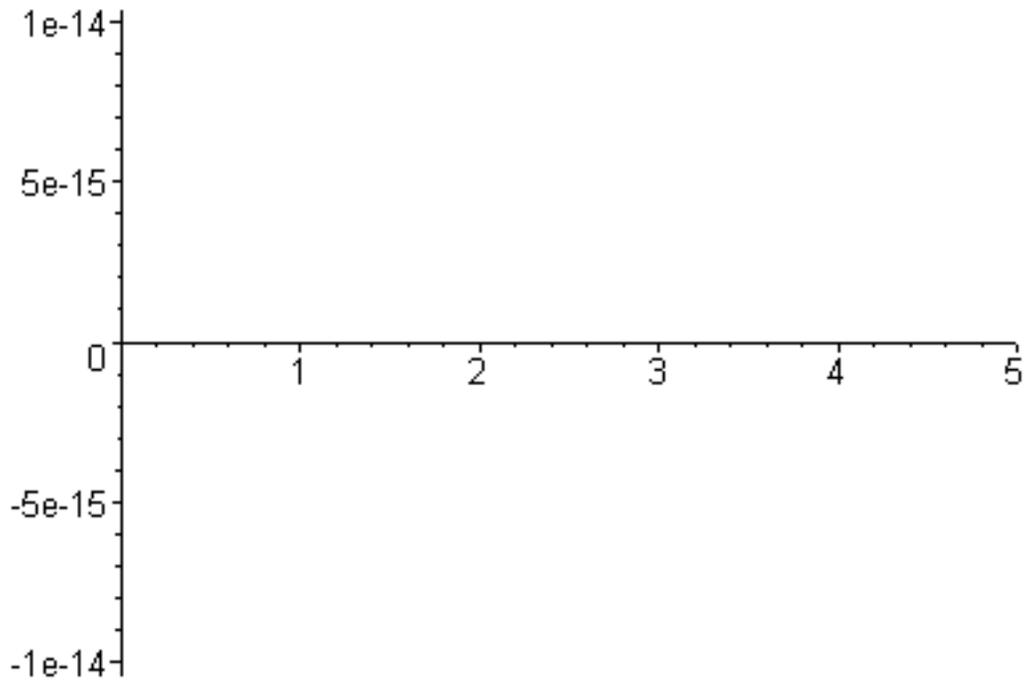
```
> plot(f(x),x=0..5);
```



```
> plot(g,x=0..5);
```



```
> plot(p,0..5);
```



The cure is to assign a value to t , like this

```
> t:=5;
```

```
t := 5
```

Now all three plot commands will work.

Error, (in plot) invalid arguments

Perhaps you are trying to plot with a variable that has been assigned a value, like this

```
> restart;
```

```
> g:=sin(x);
```

```
g := sin(x)
```

```
> x:=5;
```

```
x := 5
```

```
> plot(g(x),x=0..5);
```

```
Error, (in plot) invalid arguments
```

The cure is to undo the assignment like this (the single quotes are important)

```
> unassign('x');
```

or

```
> x:='x';
```

```
x := x
```

then try the plot again.

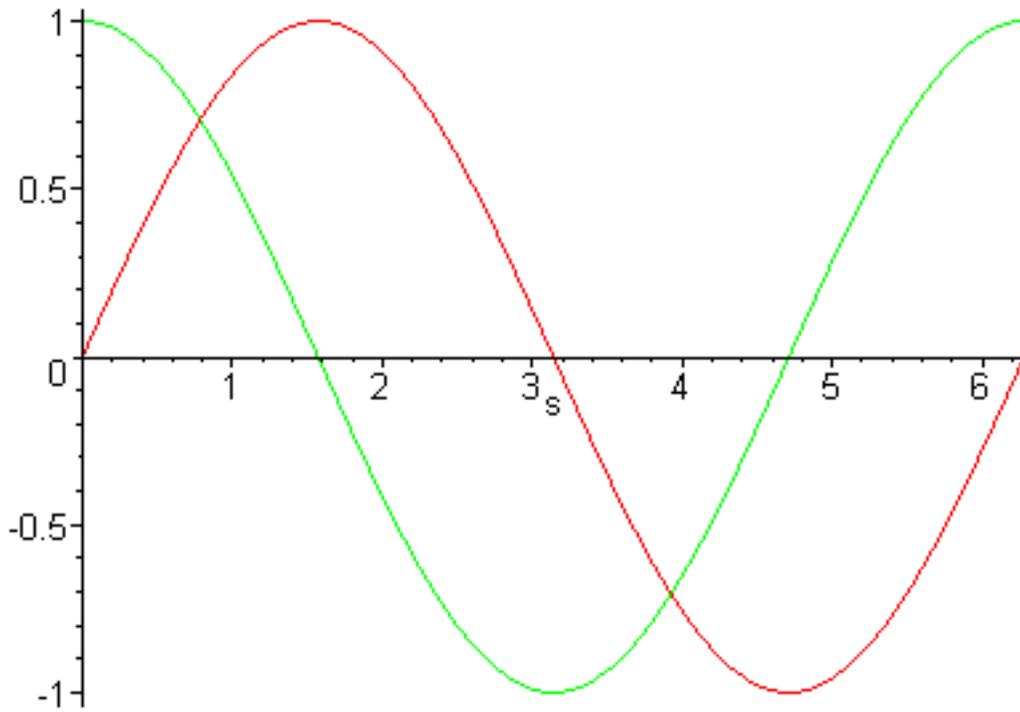
Error, (in plot/transform) cannot evaluate boolean: $-5.*z < -4$

Perhaps you have used = instead of := in giving a variable a numerical value; in any case the problem is that z doesn't have a value. This is an especially subtle error when the thing that doesn't have a value is π , which happens when you use pi instead of Pi.

Parametric plot gives two plots instead

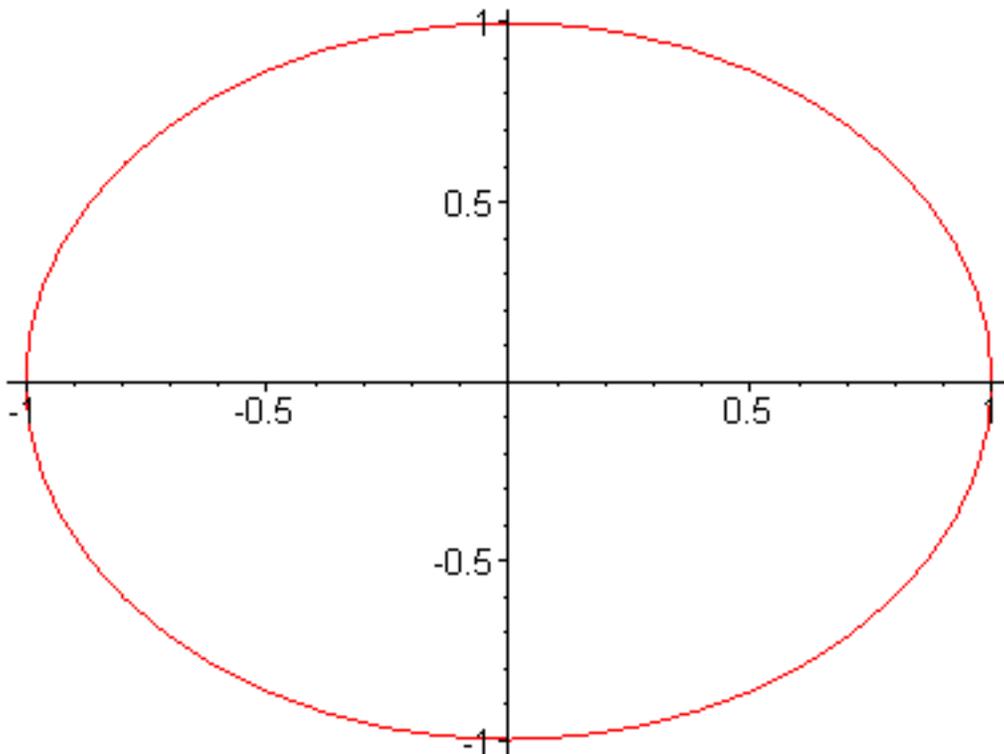
This happens when you get the square brackets in the wrong places. Here is a two-function plot

```
> plot([sin(s),cos(s)],s=0..2*Pi);
```



and here is a parametric plot

```
> plot([sin(s),cos(s),s=0..2*Pi]);
```



Look carefully at the placement of] in these two cases.

>

Function plotting fails

When you have defined a function $f(x)$, you try to plot it with `plot(f(x),x=0..5)`, and it fails, try operator form instead:

> `f:=x->fsolve(t=x/(1+exp(t)),t);`

$$f := x \rightarrow \text{fsolve} \left(t = \frac{x}{1 + e^t}, t \right)$$

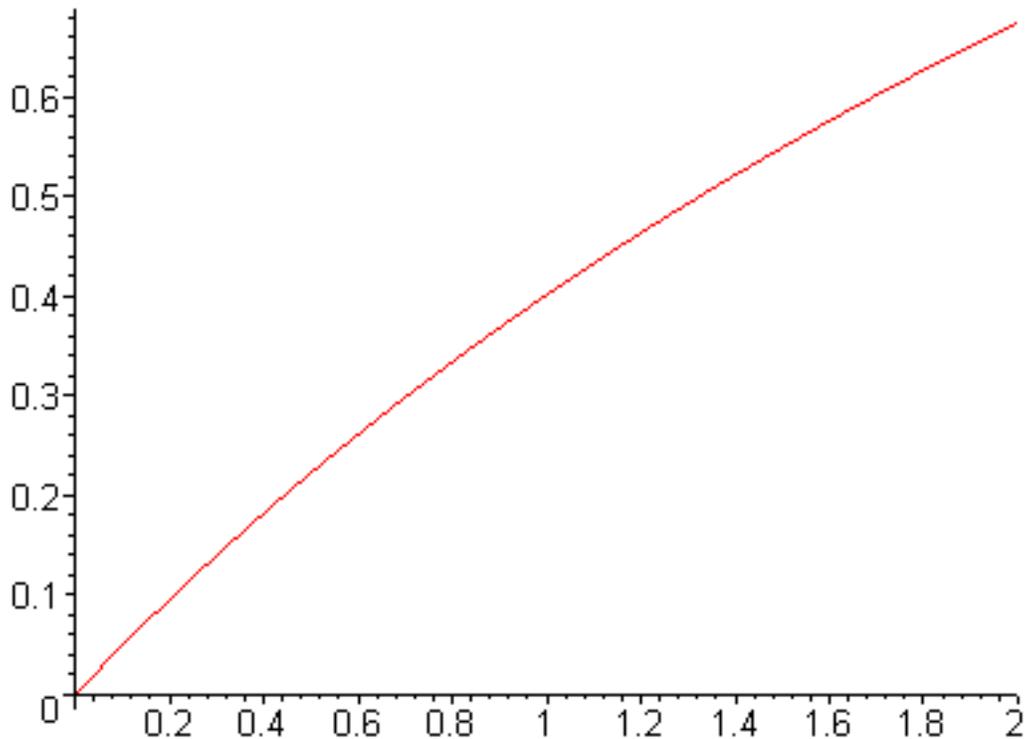
This syntax fails:

> `plot(f(x),x=0..2);`

Error, (in fsolve) x is in the equation, and is not solved for

This syntax works:

> `plot(f,0..2);`



>

D. Differentiation errors

Error, wrong number (or type) of parameters in function diff

Same thing as the unassign problem above in Plot errors, only in **diff**.

```
> restart:x:=5;
```

```
x := 5
```

```
> diff(sin(x),x);
```

Error, wrong number (or type) of parameters in function diff

You need to **unassign** x.

E. Integration errors

Error, (in int) wrong number (or type) of arguments

Same thing again as the unassign problem in Plot errors, except with **int**.

```
> restart:x:=5;
```

```
x := 5
```

```
> int(sin(x),x);
```

Error, (in int) wrong number (or type) of arguments

You need to **unassign** x.

F. Solve errors

Nothing comes back

Sometimes Maple can solve a system if the coefficients are rational numbers, but not if they are floating point numbers. So don't use $x*0.01$ -- use $x/100$ instead.

G. Dsolve errors

H. Algebra errors

Error, too many levels of recursion or Warning, recursive definition of name

You have somehow set a variable equal to a function of itself. Here is a simple way to get this error

```
> x:=1+x;
```

```
x := 6
```

but you can get it indirectly too if there is a chain of assigned variables and you close the chain on itself. It is helpful to know what variables are assigned to what, and you can find this out with the command `anames()`, unless the system is so messed up from the error that you get garbage. If this happens a `restart` is in order.

You want x to be a variable, but it is a number instead

x was previously assigned a value and you now need it to be a variable again. Here's a way to fix it.

```
> x:='x';
```

```
x := x
```

I. Miscellaneous errors

Error, (in assign) invalid arguments

Probably you tried to `unassign` a variable which was assigned a value and you didn't use single quotes, like this

```
> restart;x:=5;
```

```
x := 5
```

```
> unassign(x);
```

```
Error, (in unassign) cannot unassign `5' (argument must be assignable)
```

```
> unassign('x');
```

Note that you can also unassign with this command

```
> x:='x';
```

```
x := x
```

```
>
```

[Go to top of Debugging](#)

[Laboratory_Assignments.mws](#)

- [Laboratory assignments](#)

Laboratory assignments

This 1-credit-hour course consists of 14 computer labs to teach you how to use Maple to solve physics problems. Each lab is 3 hours long and there is no assigned work outside of this period. The text is a computer program written in Maple and called phys230.mws. You and your lab partner are assigned to read the text, execute the examples, and solve the assigned problems by using Maple, following the schedule given below. You complete each problem by showing the instructor or the TA what you have produced on your computer screen and having them record the problem as completed.

No paper will be handed in. Your final grade will be determined by how many of the assigned problems you complete and by your performance on a short oral examination at the end of the course.

Warning: some of you will be tempted to miss class and just work through the book on your own because the class feels so much like independent study. *Do not do this.* You will find yourself staring at a piece of Maple code that should work, but doesn't, and not have a clue about what is wrong. This happens in class too, but there you can raise your hand, a TA comes over and says, "You need a comma right there.", the code works, and you are underway again. By yourself this comma can take a long time to find, wasting a lot of your valuable time. So please: come to class.

Also: watch for the checkpoints listed below after Chapters 2, 5, and 9. They are there to keep you from getting behind and trying to finish the whole course in the last week.

Chapter 1 (labs 1-2)

- Open the Introduction and read it with your lab partner.
- Open the Index and look at it. Click on a topic, then figure out a way to get back to the Index.
- Open Chapter 1 and work through each section, executing all of the Maple commands and doing what the text tells you to do.
- Use the things you have learned in Chapter 1 to make a Maple worksheet that presents the following problem in text and equations, discusses how to solve it, then uses Maple commands to solve it.
- A charged disk of radius a and uniform charge density σ produces an electric field E_z along

the z -axis (where the z -axis is the line perpendicular to the disk through its center) given by the integral expression

$$E_z(z) = \frac{\int_0^a \frac{\sigma z}{(r^2 + z^2)^{\frac{3}{2}}} r dr}{2 \epsilon_0}$$

Find a simple algebraic expression for $E_z(z)$, then use $\sigma = 1 \text{ e-6 } \frac{\text{C}}{\text{m}^2}$, $\epsilon_0 = 8.854 \text{ e-12}$

$\frac{\text{Nm}^2}{\text{C}}$, and $a = 1 \text{ mm}$ to make a plot of $E_z(z)$ from $z = -5a$ to $z = 5a$.

(f) In Example 15.13 of Serway the Venturi tube is discussed. Bernoulli's equation

$$P_1 + \frac{\rho v_1^2}{2} = P_2 + \frac{\rho v_2^2}{2}$$

and the equation of continuity

$$A_1 v_1 = A_2 v_2$$

are solved simultaneously to obtain the following formula for v_2 :

$$v_2 = A_1 \sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}$$

Use Maple's **solve** command to obtain this result.

Chapters 2 and 9 (labs 3-4)

(a) Go to Chapter 9 and study the symbolic algebra commands listed there, running the examples as you go. Don't work through the long exercise at the end, but just try to become familiar with the commands and what they do. As you work the other problems in this course you will probably refer to this chapter often as you try to talk Maple into giving you results in the form you want.

(b) Work through the sections of Chapter 2 from x-y Plotting to Plotting Data, skipping the advanced topics. Work the following problems and show your work to your TA.

2.1, 2.2, 2.3, 2.6, 2.7, 2.8, and the Plotting Data example.

(c) Work through the sections of Chapter 2 from Parametric Plots to 3-D Plotting and do the following problems:

2.10, 2.12, 2.13, 2.14, 2.16, 2.18, 2.20, 2.24, and work through and experiment with the commands in the 3D Plotting section. Especially become familiar with the items available on the toolbar at the top of the plot frame.

(d) Not required, but a good idea: If you have time, do the advanced material on wave packets in the section on animations and do problems 2.21-23.

CHECKPOINT 1:

Chapters 1 and 2 must be finished now; no late work accepted after the end of the 4th laboratory period.

Chapter 3 (labs 5-7)

(a) Work through Limits, Differentiation, and part of Integration in Chapter 3, up through Elementary Integrals. Show your TA your work on Problems 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, and 3.11.

(b) Work through the rest of Integration in Chapter 3, and show to your TA Problems 3.12, 3.13, 3.14, 3.16, and 3.17.

(c) Work through Series Expansions and Sums. Do Problems 3.19, 3.20, 3.21, 3.22, 3.23, 3.24, 3.25, and 3.26.

Chapter 4 (lab 8)

Work through Chapter 4 on Complex Analysis, doing Problems 4.1, 4.2, 4.3, 4.4, 4.5, 4.7, 4.8, 4.11, 4.12, and 4.13.

Chapter 5 (lab 9)

Work through Chapter 5 on Linear Algebra, doing Problems 5.1, 5.2, 5.3, 5.4, 5.5, and 5.6.

CHECKPOINT 2:

Chapters 3-5 must be finished now; no late work accepted after the end of the 9th laboratory period.

Chapter 6 (lab 10)

(a) Work through Chapter 6 on Solving Equations, doing problems 6.1, 6.2, 6.4, and 6.5.

(b) Find all of the zeros of the Bessel function derivative $\frac{\partial}{\partial x} J_2(x)$ between 0 and 100.

Load them into a column vector $\alpha_{\mathcal{N}}$. You can do this problem very compactly by using the **seq** command with **fsolves** inside it, remembering that the zeros of Bessel functions are separated by about π . A plot of the derivative expression will immediately show you that there is a root at $x = 0$, but

Maple will have trouble finding it because of the $\frac{1}{x}$ that appears in the derivative formula. Just load

this simple root by hand and let Maple find the rest.

Chapter 8 (labs 11-12)

(a) Work through the first two sections of Chapter 8 on Procedures, Loops and Logic, and do problems

8.1, 8.2, 8.3, 8.4, 8.5, 8.6, and 8.7.

(b) Work through the last two sections of Chapter 8, doing problems 8.8, 8.9, 8.10, 8.13, and 8.14.

Chapter 9 (labs 13-14)

(a) Look at Chapter 9 again and do Problems 9.1 and 9.2. Then do one of the three problems 9.3, 9.4, or 9.5.

(b) Come see the instructor and take a 20-minute oral/computer exam to show that you have a working knowledge of Maple.

(c) Catch up on any late lab work and work on the three homework problems from your other physics or math classes. Pass them off to the TA.

CHECKPOINT 3:

Chapters 6, 8-9 must be finished now; no late work accepted after the last lab period.

[Go to top of section](#)