EL582/BE620 --- Medical Imaging - I

Computed Tomography (part II)

Yao Wang
Polytechnic University, Brooklyn, NY 11201

Based on J. L. Prince and J. M. Links, Medical Imaging Signals and Systems, and lecture notes by Prince. Figures are from the textbook.

Last Lecture

- Instrumentation
  - CT Generations
  - X-ray source and collimation
  - CT detectors
- Image Formation
  - Line integrals
  - Parallel Ray Reconstruction
    - Radon transform
    - Back projection
    - Filtered backprojection
    - Convolution backprojection
    - Implementation issues

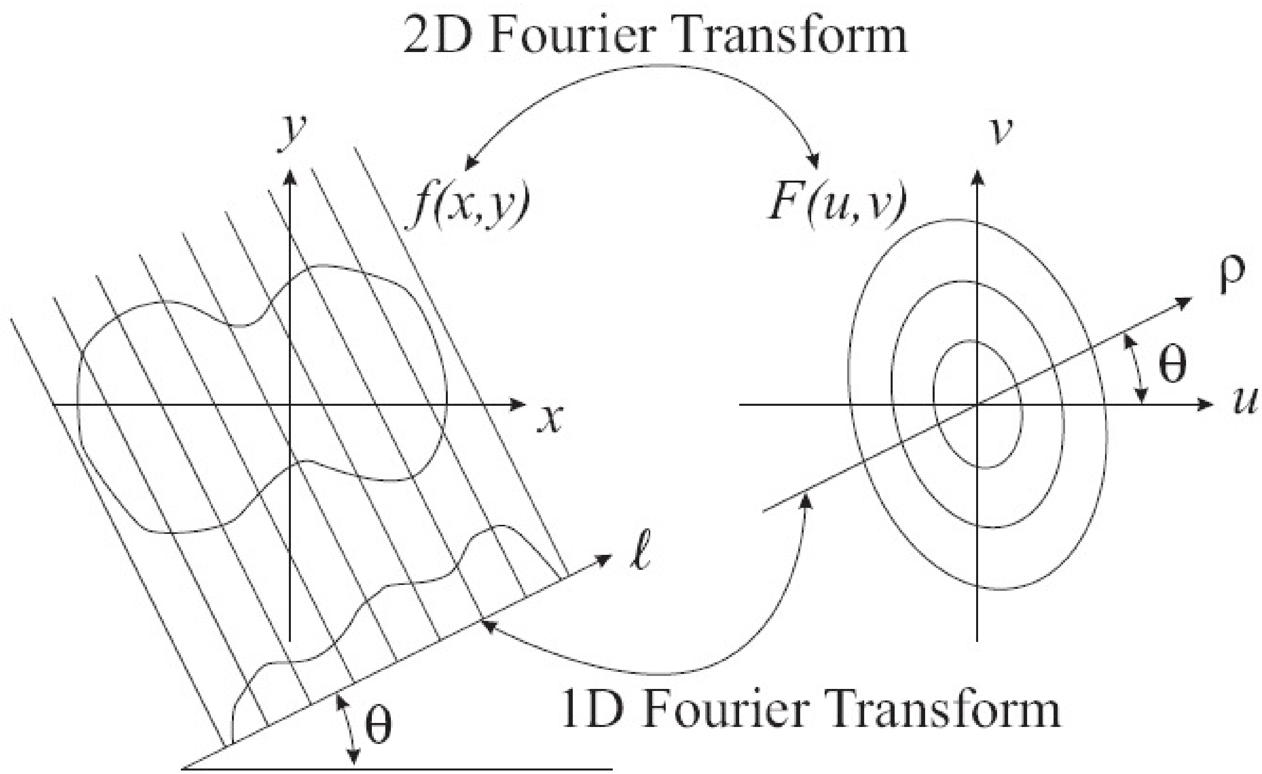
This Lecture

- Review of Parallel Ray Projection and Reconstruction
- Practical implementation with samples
- Fan Beam Reconstruction
- Signal to Noise in CT

Review: Projection Slice Theorem

- Projection Slice theorem
  - The Fourier Transform of a projection at angle θ is a line in the Fourier transform of the image at the same angle.

$$G(\varrho, \theta) = F(\varrho \cos \theta, \varrho \sin \theta)$$



Reconstruction Algorithm for Parallel Projections

- Backprojection:
  - Backprojection of each projection
  - Sum

$$f_b(x, y) = \int_0^{\pi} [g(\ell, \theta)]_{\ell = x\cos\theta + y\sin\theta} \, d\theta$$

- Filtered backprojection:
  - FT of each projection
  - Filtering each projection in frequency domain
  - Inverse FT
  - Backprojection
  - Sum

$$f(x, y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} |\varrho| G(\varrho, \theta) e^{+j2\pi\varrho\ell} d\varrho \right]_{\ell = x\cos\theta + y\sin\theta} d\theta$$

- Convolution backprojection
  - Convolve each projection with the ramp filter
  - Backprojection
  - Sum

$$f(x, y) = \int_0^{\pi} [c(\ell) * g(\ell, \theta)]_{\ell = x\cos\theta + y\sin\theta} \, d\theta$$

Practical Implementation

- Projections $g(l, \theta)$ are only measured at finite intervals
  - $l = n\tau$;
  - τ chosen based on maximum frequency in $G(\rho, \theta)$, W
    - $1/\tau >= 2W$ or $\tau <= 1/2W$ (Nyquist Sampling Theorem)
    - W can be estimated by the number of cycles/cm in the projection direction in the most detailed area in the slice to be scanned
- For filtered backprojection:
  - Fourier transform $G(\rho, \theta)$ is obtained via FFT using samples $g(n\tau, \theta)$
  - If N sample are taken, 2N point FFT is taken by zero padding $g(n\tau, \theta)$
    - *Recall convolving two signals of length N leads to a single of length 2N-1*
- For convolution backprojection
  - The ramp-filter is sampled at $l = n\tau$
  - Sampled Ram-Lak Filter

$$c(n) = \begin{cases} 1/4\tau^2; & n = 0 \\ -1/(n\pi\tau)^2; & n = odd \\ 0; & n = even \end{cases}$$

The Ram-Lak Filter (from [Kak&Slaney])

$$H(w) = |w| b_w(w)$$

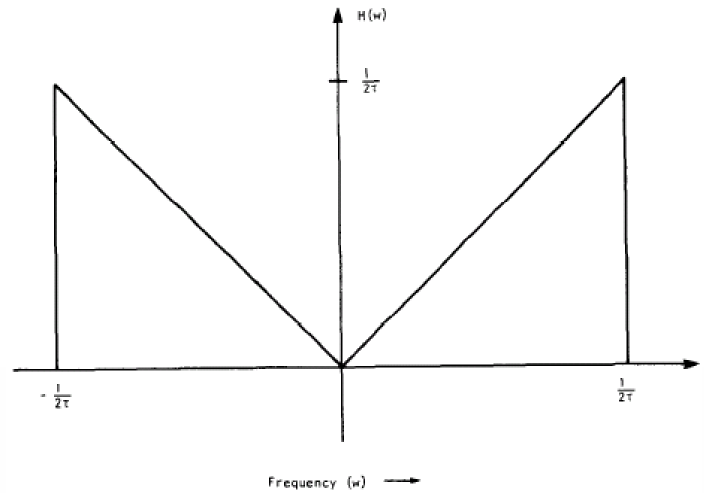
$$b_W(w) = \begin{cases} 1 & |w| < W \\ 0 & \text{otherwise.} \end{cases}$$

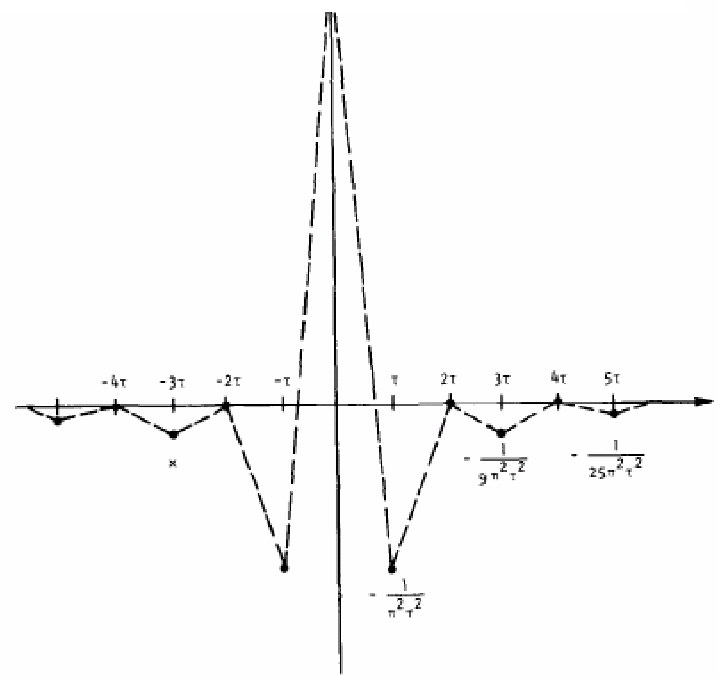
$$W = \frac{1}{2\tau} \text{ cycles/cm.}$$

$$h(t) = \int_{-\infty}^{\infty} H(w) e^{+j2\pi w t}\, dw$$

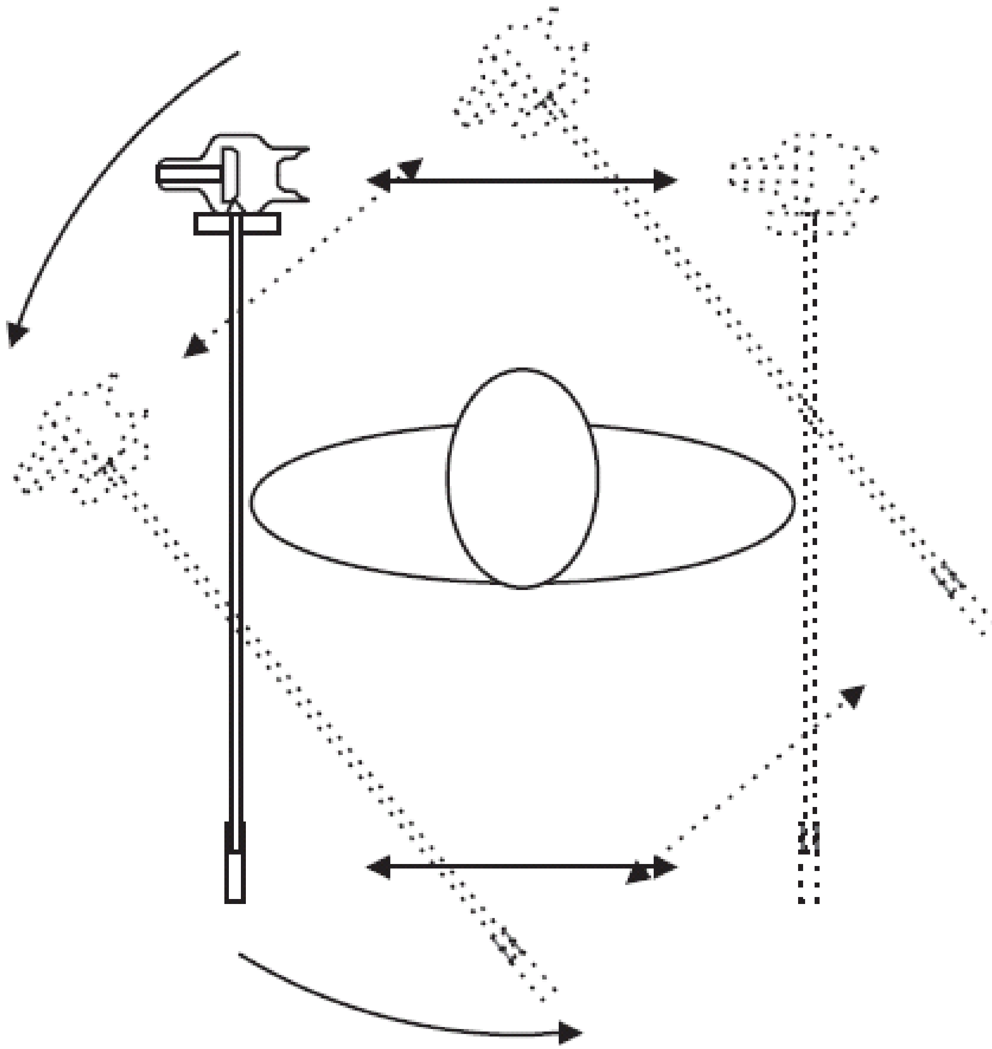
$$= \frac{1}{2\tau^2} \frac{\sin 2\pi t/2\tau}{2\pi t/2\tau} - \frac{1}{4\tau^2} \left(\frac{\sin \pi t/2\tau}{\pi t/2\tau} \right)^2$$

$$h(n\tau) = \begin{cases} 1/4\tau^2, & n = 0 \\ 0, & n \text{ even} \\ -\dfrac{1}{n^2 \pi^2 \tau^2}, & n \text{ odd.} \end{cases}$$

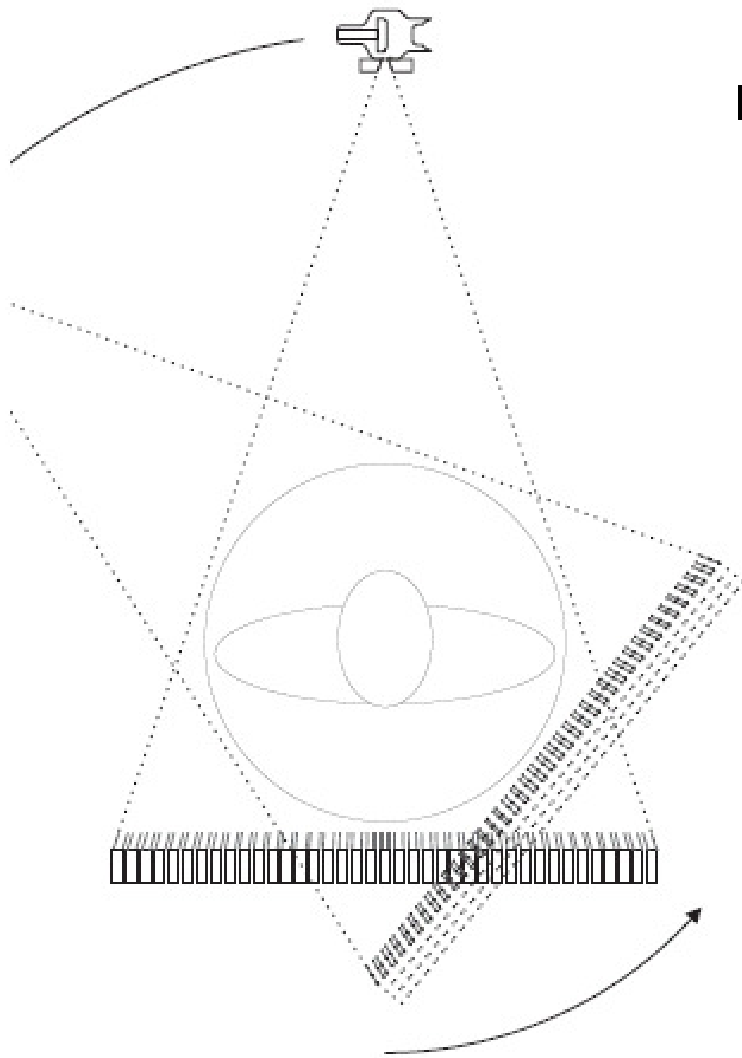




1st Generation CT: Parallel Projections

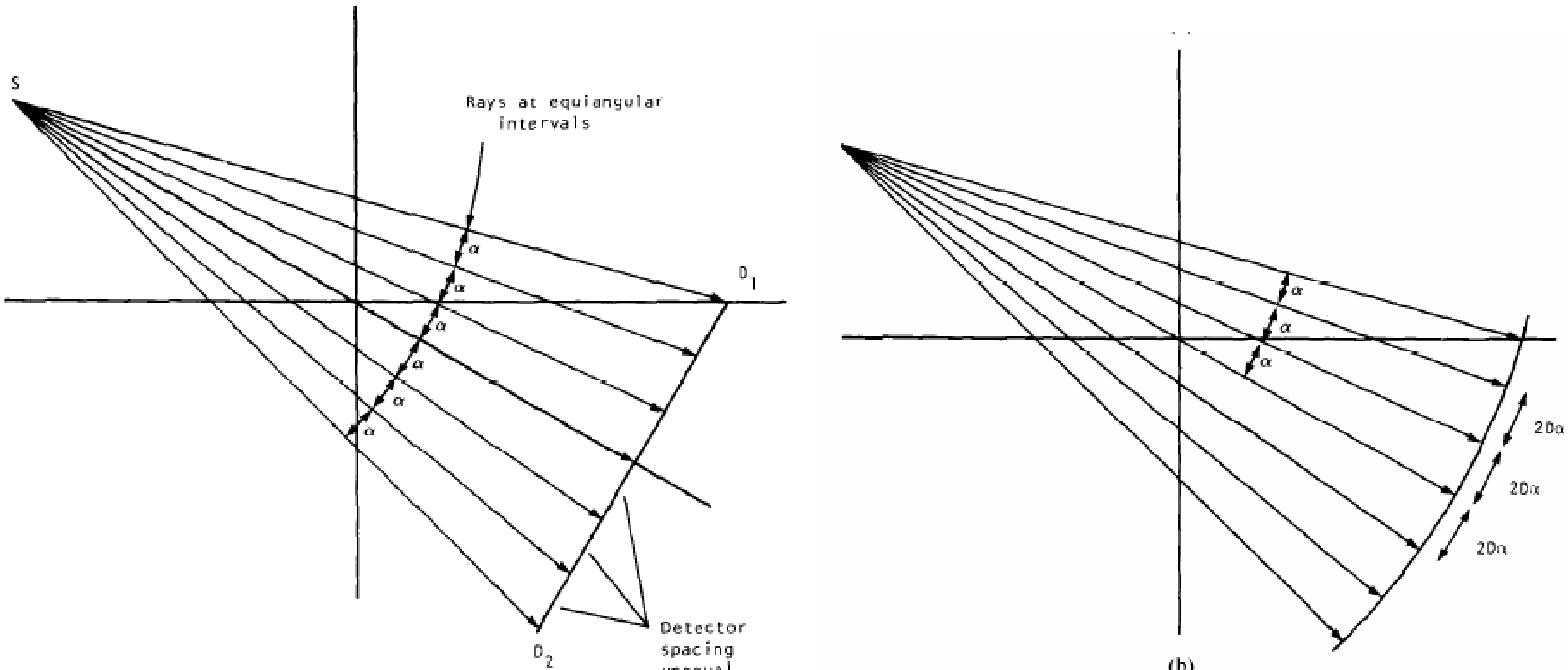


3G: Fan Beam



Much faster than 2G

Fan Beam: Equiangular Ray

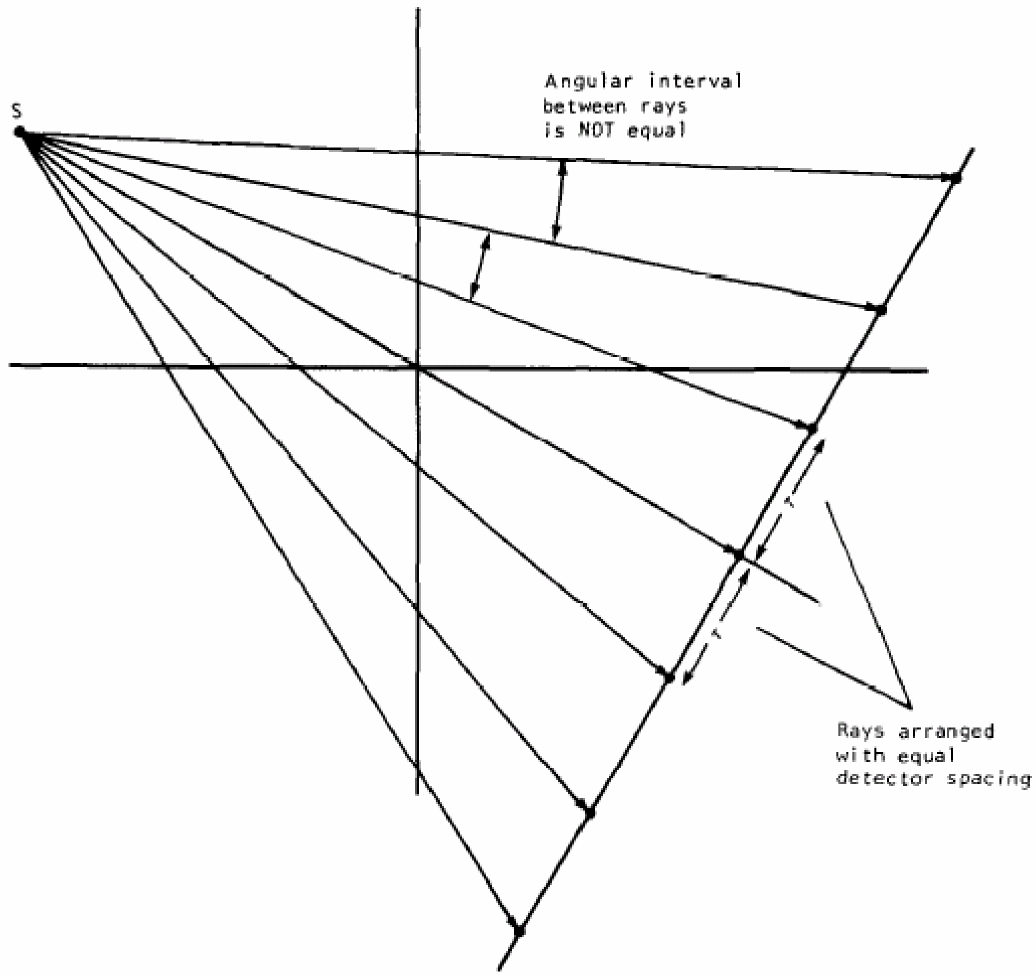


(a)

(b)

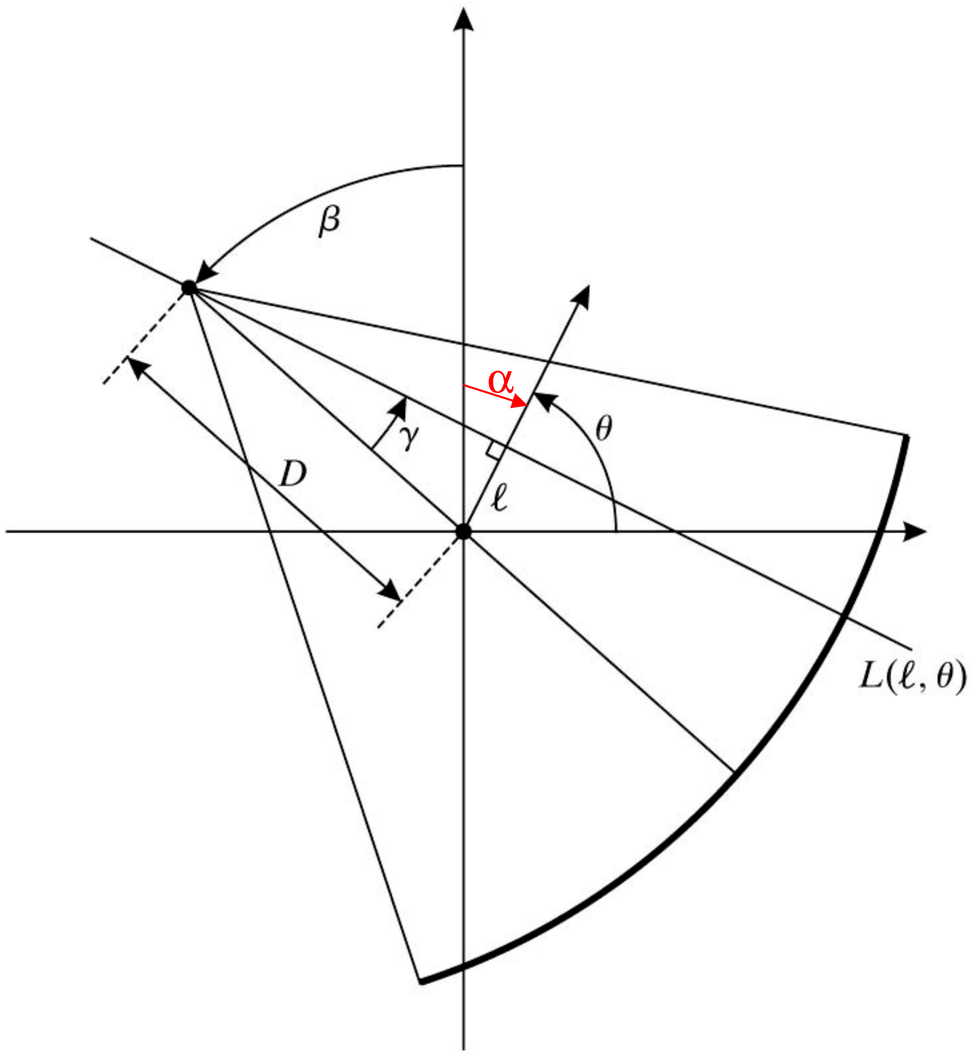
We will focus on the equiangular detector setting on the right in this lecture.

Fan Beam: Equidistant Ray



We will skip the discussion on reconstruction from equidistant Ray.
Details can be found at [Kak&Slaney]

Equiangular Ray Projection



Source location is described by (β, D)

D is typically fixed, β varies to provide a large view angle.

To provide complete view, $\beta \in (0, 2\pi)$

For a given source with angle β, γ specifies the detector position or the projection line.

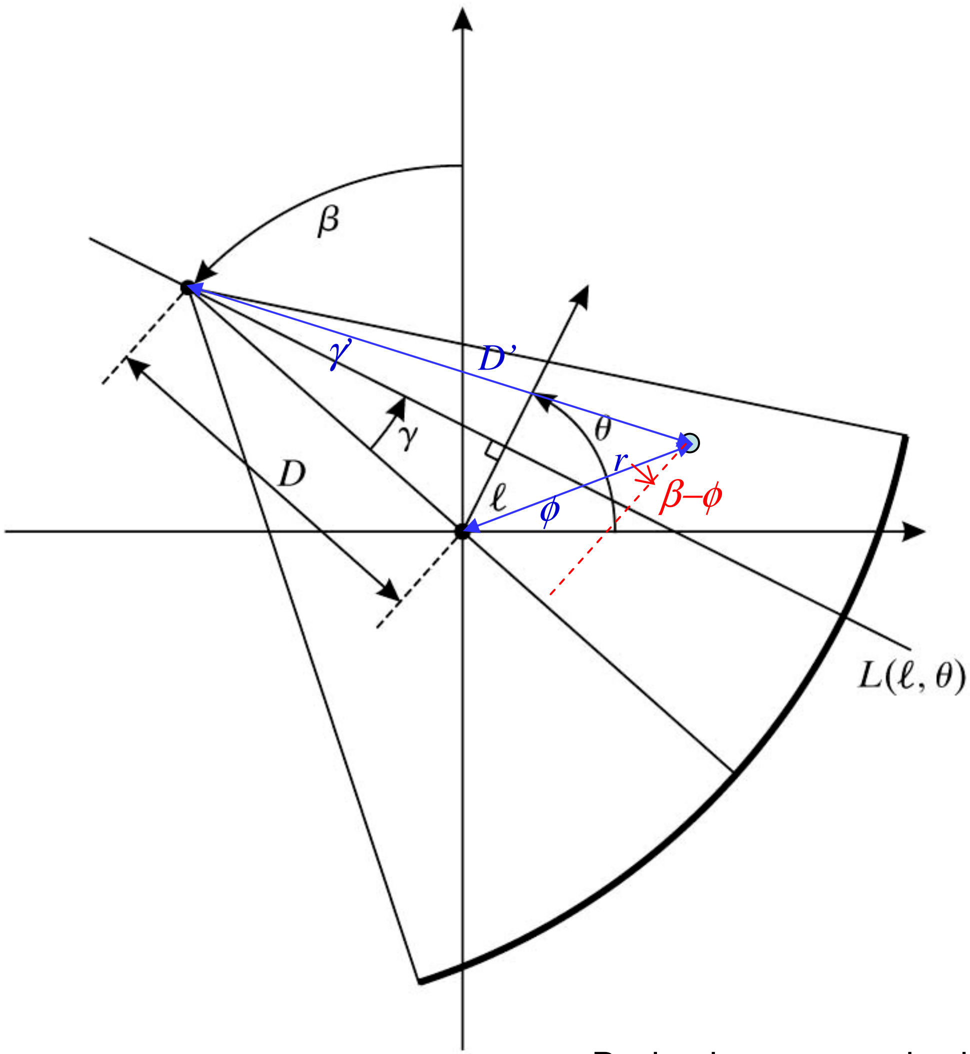
For each β, γ varies over a range $(-\gamma_m, \gamma_m)$

(D, β, γ) completely specifies the line of projection:

$\theta = \beta + \gamma, l = D\sin(\gamma)$

Instead of $g(l, \theta)$, we can use $p(\gamma, \beta)$ to represent a projection

Equiangular Ray Reconstruction



Reconstructed image is represented in the polar coordinate using (r,ϕ).

The relative position of a pixel at (r,ϕ) to the source at (D,β) is specified by (D',γ'):

$$D'^2(r,\phi) = \left(D + r\sin(\beta-\phi)\right)^2 + \left(r\cos(\beta-\phi)\right)^2$$

$$\tan\gamma'(r,\phi) = \left(r\cos(\beta-\phi)\right) \Big/ \left(D + r\sin(\beta-\phi)\right)$$

Reconstruction formular:

$$f(r,\phi) = \int_0^{2\pi} \frac{1}{(D')^2} q(\gamma',\beta)\,d\beta$$

Weighted backprojection

$$q(\gamma,\beta) = p'(\gamma,\beta) * c_f(\gamma)$$

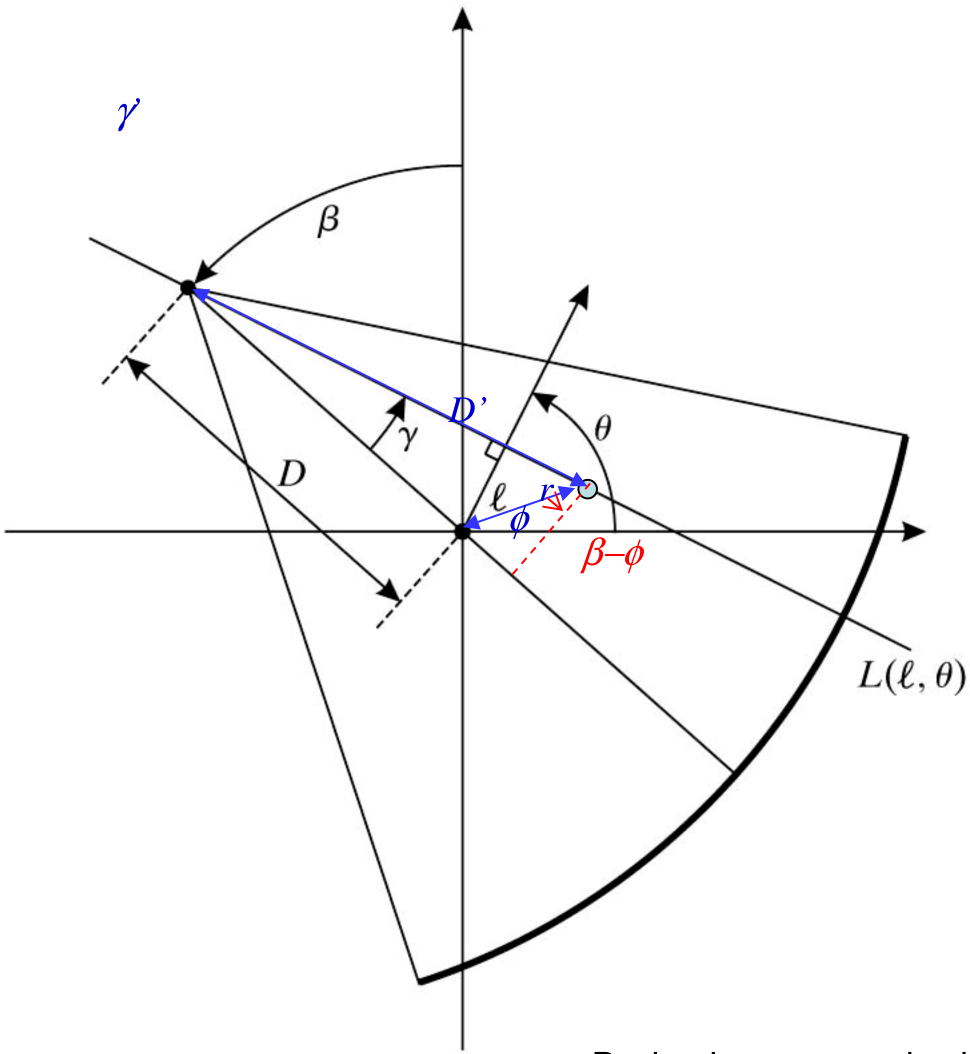
$$p'(\gamma,\beta) = p(\gamma,\beta) D\cos(\gamma)$$

$$c_f(\gamma) = \frac{1}{2}\left(\frac{\gamma}{\sin\gamma}\right)^2 c(\gamma)$$

$c(\gamma)$ is the ramp filter used in parallel projection.

Derivation not required for this class. Detail can be found at [Kak&Slaney]. Note typos in [Prince&Links]

Equiangular Ray Reconstruction



Reconstructed image is represented in the polar coordinate using (r,ϕ).

The relative position of a pixel at (r,ϕ) to the source at (D,β) is specified by (D',γ):

$$D'^2(r,\phi) = \left(D + r\sin(\beta-\phi)\right)^2 + \left(r\cos(\beta-\phi)\right)^2$$

$$\tan\gamma(r,\phi) = \left(r\cos(\beta-\phi)\right) \Big/ \left(D + r\sin(\beta-\phi)\right)$$

Reconstruction formular:

$$f(r,\phi) = \int_0^{2\pi} \frac{1}{(D')^2} q(\gamma,\beta)\, d\beta$$

Weighted backprojection

$$q(\gamma,\beta) = p'(\gamma,\beta) * c_f(\gamma)$$

$$p'(\gamma,\beta) = p(\gamma,\beta) D\cos(\gamma)$$

$$c_f(\gamma) = \frac{1}{2}\left(\frac{\gamma}{\sin\gamma}\right)^2 c(\gamma)$$

$c(\gamma)$ is the ramp filter used in parallel projection.

Derivation not required for this class. Detail can be found at [Kak&Slaney]. Note typos in [Prince&Links]

Typos in [Prince&Links]

- P. 207, Eq. (6.38), change to

$$c(D'\sin\gamma) = \left(\frac{\gamma}{D'\sin\gamma}\right)^2 c(\gamma)$$

- Eq. (6.39) change to $$c_f(\gamma) = \frac{1}{2}\left(\frac{\gamma}{\sin\gamma}\right)^2 c(\gamma)$$

- Eq. (6.40),(6.41)

$$p(\gamma,\beta) \to p'(\gamma,\beta)$$
$$p'(\gamma,\beta) = p(\gamma,\beta) D\cos(\gamma)$$
$$q(\gamma,\beta) = p'(\gamma,\beta) * c_f(\gamma)$$

Practical Implementation

- Projections $P(\gamma, \beta)$ are only measured at finite intervals
  - $\gamma = n\alpha$;
  - α chosen based on maximum frequency in γ direction, W
    - $1/\alpha >= 2W$ or $\alpha <= 1/2W$
- For convolution backprojection
  - The filter $c_f(\gamma)$ is sampled at $\gamma = n\alpha$
  - Sampled Filter $g(n\alpha)$
- For backprojection
  - For given (r,ϕ), determine (D',γ)
  - Use interpolation to determine $q(\gamma,\beta)$ from known values at $\gamma = n\alpha$

$$g(n\alpha) = \begin{cases} \dfrac{1}{8\alpha^2}, & n = 0 \\ 0, & n \text{ is even} \\ -1/2 \left(\dfrac{\alpha}{\pi\alpha \sin n\alpha}\right)^2, & n \text{ is odd.} \end{cases}$$

$$D'^2(r,\phi) = (D + r\sin(\beta-\phi))^2 + (r\cos(\beta-\phi))^2$$

$$\tan\gamma(r,\phi) = (r\cos(\beta-\phi)) \big/ (D + r\sin(\beta-\phi))$$

$$f(r,\phi) = \int_0^{2\pi} \frac{1}{(D')^2} q(\gamma,\beta)\, d\beta$$

$$q(\gamma,\beta) = p'(\gamma,\beta) * c_f(\gamma)$$

$$p'(\gamma,\beta) = p(\gamma,\beta) D\cos(\gamma)$$

Matlab Functions for Fan Beam CT

- Relevant functions:
  - fanbeam(), ifanbeam()

CT Quality Evaluation

- Blurring Effect
- SNR

Effect of Area Detector

- Practical detector integrates the detected photons over an area
- Mathematically, the detector can be characterized by an indicator function *s(l) (aka impulse response)*
- The measured projection $g'(l,\theta)$ is related to "real" projection $g(l,\theta)$ by
  - $g'(l,\theta) = g(l,\theta) * s(l)$
  - $G'(\rho,\theta) = G(\rho,\theta)\, S(\rho)$

$$S(\varrho) = \mathcal{F}\{s(\ell)\}$$

Windowing Function

- Recall that the ideal filter *c(ρ)* is typically modified by a window function *W(ρ)*

- Overall Effect

$$\hat{f}(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} G(\varrho,\theta) S(\varrho) W(\varrho) |\varrho| e^{j2\pi\varrho\ell} \, d\varrho \right]_{\ell = x\cos\theta + y\sin\theta} d\theta$$

$\hat{f}(x,y)$ can be thought of as the reconstructed image from the projection $\hat{g}(l,\theta)$, whose Fourier transform is

$$\hat{G}(\rho,\theta) = G(\rho,\theta) S(\rho) W(\rho) \Leftrightarrow \hat{g}(l,\theta) = g(l,\theta) * s(l) * w(l)$$

Blurred Projection

- Blurry projection:

$$\begin{aligned}\hat{g}(\ell,\theta) &= g(\ell,\theta) * s(\ell) * w(\ell) \\ &= g(\ell,\theta) * \tilde{h}(\ell)\end{aligned}$$

- Radon transform convolution theorem

$$\mathcal{R}\{f *_2 h\} = \mathcal{R}\{f\} *_1 \mathcal{R}\{h\}$$

- Leads to

h(x,y): PSF of the blurring

$$\hat{f}(x,y) = f(x,y) * \mathcal{R}^{-1}\{\tilde{h}(\ell)\}$$

Circular Symmetry of Blurring

- CT image blurred by convolution kernel

$$h(x,y) = \mathcal{R}^{-1}\{\tilde{h}(\ell)\}$$

- Fourier transform of $\tilde{h}(\ell)$

$$\tilde{H}(\varrho) = \mathcal{F}_1\{\tilde{h}(\ell)\} = S(\varrho)W(\varrho)$$

which is independent of θ.

- Therefore, $H(u,v)$ is circularly symmetric

$$H(q) = \mathcal{F}_2\{h(x,y)\} = S(q)W(q)$$

PSF given by Hankel Transform

- PSF is circularly symmetric and given by

$$\mathrm{h}(r) = \mathcal{H}^{-1}\{S(\varrho)W(\varrho)\}$$

- Reconstructed image given by

$$\hat{f}(x,y) = f(x,y) * \mathrm{h}(r)$$

$$r^2 = x^2 + y^2$$

Circularly Symmetric Functions and Hankel Transform

- Circularly symmetric:
  - f(x,y) = f(r), only depends on the distance to the origin, not angle
- Fourier transform of circularly symmetric function is also circularly symmetric
  - F(u,v)=F(ρ)

Let $x = r\cos\phi,\ y = r\sin\phi; u = \rho\cos\theta, v = \rho\sin\theta$

$$F(\rho,\theta) = \iint f(r,\phi)\exp\{-j2\pi(r\rho\cos\phi\cos\theta + r\rho\sin\phi\sin\theta)\}rdrd\phi$$

$$= \iint f(r,\phi)\exp\{-j2\pi r\rho\cos(\phi-\theta)\}rdrd\phi$$

If $f(x,y) = f(r)$

$$F(\rho,\theta) = \int\left\{\int \exp\{-j2\pi r\rho\cos(\phi-\theta)\}d\phi\right\}f(r)rdr = 2\pi\int_0^\infty f(r)J_0(2\pi\rho r)rdr = F(\rho)$$

Hankel Transform

$$F(\rho) = 2\pi\int_0^\infty f(r)J_0(2\pi\rho r)rdr; \quad J_0(r) = \frac{1}{\pi}\int_0^\pi \cos(r\sin\phi)d\phi$$

Common Transform pairs

- See Table 2.3

$$Fourier\left\{e^{-\pi(x^2+y^2)}\right\}=e^{-\pi(u^2+v^2)}$$

$$=> \text{Hankel}\left\{e^{-\pi r^2}\right\}=e^{-\pi\rho^2}$$

$$\text{Hankel}\{\sin c(r)\}=\frac{2rect(q)}{\pi\sqrt{1-4q^2}}$$

- Scaling property $$\text{Hankel}\{\text{f(ar)}\}=\frac{1}{a^2}F(q/a)$$

- Duality: If h(r) <-> H(ρ), then H(r)<->h(ρ)

Derivation of Hankel transform pairs are not required. But you should be able to use given transform pairs, to determine the blur function.

Example

- Example 6.5 in [Prince&Links]
  - Detector: rectangular detector with width d
    - S(l)=rect(l/d)
  - Rectangular window function
    - $W(\rho)=rect(\rho/2\rho_o)$; $\rho_o>>1/d$
- Solution
  - S(l)=rect(l/d) <-> $S(\rho)=d\ sinc(d\rho)$
  - $\rho_o>>1/d$ ->
  - $H(\rho)=S(\rho)\ W(\rho) \sim= S(\rho) = d sinc(d\rho)$ (Hankel transform of h(r))

$$h(r) = \text{inverse Hankel}\{H(\rho)\}$$

$$\text{Hankel}\{\text{sinc}(r)\} = \frac{2rect(\rho)}{\pi\sqrt{1-4\rho^2}}$$

$$\text{Hankel}\{\text{f(ar)}\} = \frac{1}{a^2} F(\rho / a)$$

Using duality property

$$h(r) = d\frac{2rect(r/d)}{\pi\sqrt{d^2-4r^2}}$$

Illustrate and explain h(r)

Noise in CT Measurement

- Basic measurement is:

$$g_{ij} = -\ln\left(\frac{N_{ij}}{N_0}\right)$$

  - line L_{ij}
  - angle i
  - position j
- Noise is "in" Poisson random variable N_{ij}
  - mean $\bar{N}_{ij}$
  - variance $\bar{N}_{ij}$

$$\Pr\{N_{ij} = k\} = \frac{a^k}{k!} e^{-a}; \quad k = 0, 1, \ldots$$

$$E\{N_{ij} = k\} = a$$

$$Var\{N_{ij} = k\} = a$$

What about the measured projection

- It follows that g_{ij} is a random variable

$$\bar{g}_{ij} \approx \ln\left(\frac{N_0}{\bar{N}_{ij}}\right)$$

$$\mathrm{Var}(g_{ij}) \approx \frac{1}{\bar{N}_{ij}}$$

- $\hat{\mu}(x, y)$ is approximate reconstruction
- It follows that $\hat{\mu}(x, y)$ is a random variable
- What are the mean and variance of $\hat{\mu}$?

CBP Approximation

- Convolution backprojection (CBP):

$$\mu(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} g(\ell,\theta)c(x\cos\theta + y\sin\theta - \ell)d\ell d\theta$$

- Approximations:
  - M angles; $\Delta\theta = \pi/M$
  - $N+1$ detectors; $\Delta\ell = T$
  - $\tilde{c}(\ell) \approx c(\ell)$
- Discrete CBP:

$$\hat{\mu}(x,y) = \left(\frac{\pi}{M}\right)\sum_{j=1}^{M} T \sum_{i=-N/2}^{N/2} g(iT, j\pi/M)\tilde{c}(x\cos\theta_j + y\sin\theta_j - iT)$$

Definitions and Assumptions

- $\bar{N}_{ij}$ is mean for i-th detector and j-th angle
- N_{ij} is independent for different measurements
- $\bar{N}_{ij} = \bar{N}$, an "object uniformity" assumption
- $\tilde{c}(\ell)$ is created using rectangular window $W(\varrho)$ with cutoff ϱ_0.

g_{ij} are independent because N_{ij} are independent
Deriving mean and variance of $\mu(x,y)$ based on the independence assumption
See [Prince&Links] for derivation

- $\mathrm{Mean}(\hat{\mu})$ is desired result
- $\mathrm{Var}(\hat{\mu}) = \sigma_{\hat{\mu}}^2$ is inaccuracy

$$\sigma_{\hat{\mu}}^2 \approx \frac{2\pi^2}{3} \varrho_0^3 \frac{1}{M} \frac{1}{\bar{N}/T}$$

- Be cautious on conclusions: not all variables are independent in a real physical system

- Variance increases with ρ_0 (cut-off freq. of filter), and T (detector spacing), decreases with M (number of angles),\bar N (or N_0) (x-ray intensity)

SNR of the Reconstructed Image

- Definition (usual)

C: fractional change of μ from \bar μ

$$\mathrm{SNR} = \frac{C\bar{\mu}}{\sigma_{\hat{\mu}}}$$

- After substitution:

$$\mathrm{SNR} = \frac{C\bar{\mu}}{\pi}\sqrt{\frac{3M}{2\varrho_0^3}\frac{\bar{N}}{T}}$$

SNR in a good design

- What should ϱ_0 be?
- Let dectector width $= w$
- ϱ_0 should be anti-aliasing filter:

$$\varrho_0 = \frac{k}{w} \quad \text{where } k \approx 1$$

- In 3G scanner $w = T$
- Then

$$\text{SNR} \approx 0.4kC\bar{\mu}w\sqrt{\bar{N}M}$$

SNR in Fan Beam

- Definitions:
  - $\bar{N}_f$ is mean photon count per fan
  - D is number of detectors
  - L is length of detector array
- Then

D
)

$$\mathrm{SNR} \approx 0.4kC\bar{\mu}L\sqrt{\frac{\bar{N}_f M}{D^3}}$$

SNR decreases as D increases.
Reason: Convolution of the projection reading with the ramp filter couples the noise between detectors, and effectively increases the noise as the number of detector increases
But larger D is desired to obtain a good resolution.

Rule of Thumb

- Variables:
  - D is number of detectors
  - M is number of angles
  - J^2 is number of pixels in image
- Very approximate "rule":

$$D \approx M \approx J$$

- Typical numbers:

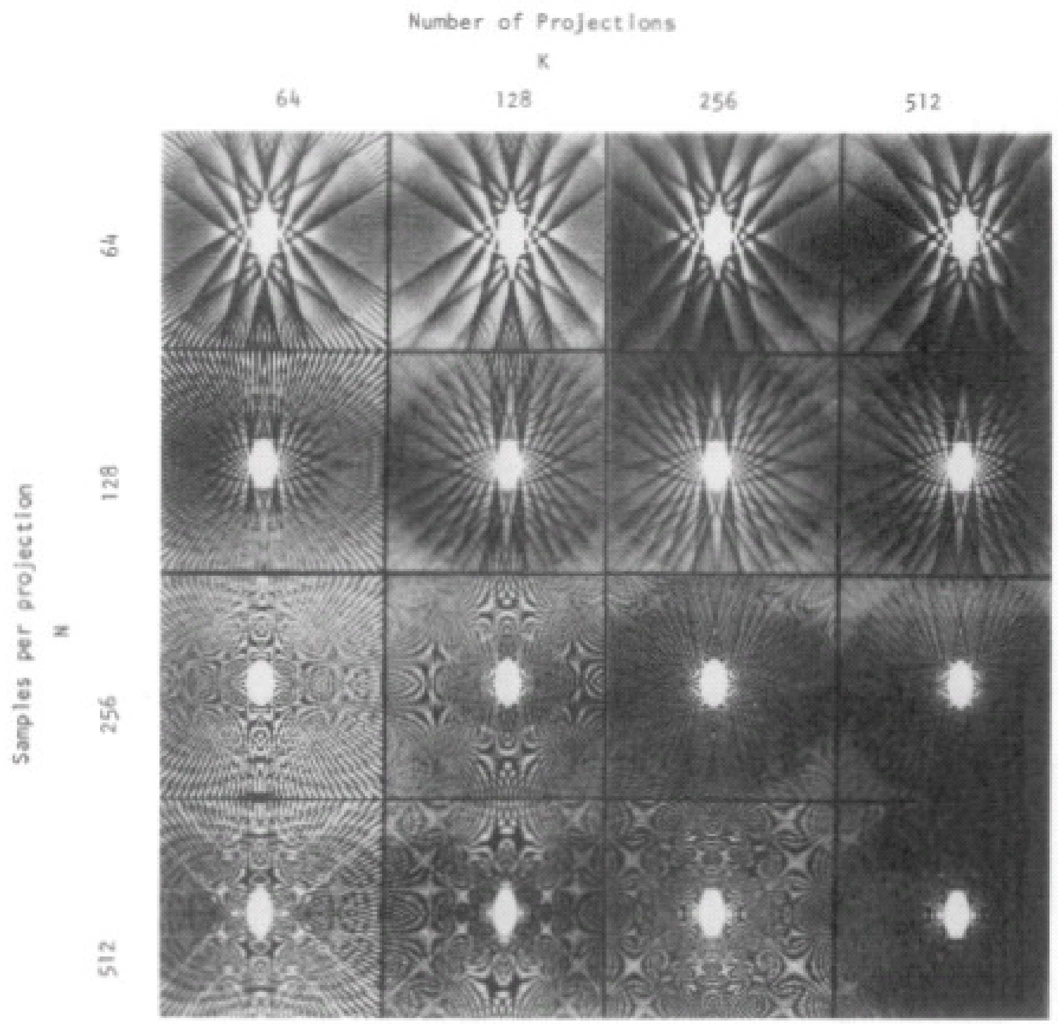
$$\text{Lo: } D \approx 700 \quad M \approx 1,000 \quad J \approx 512$$

$$\text{Hi: } D \approx 900 \quad M \approx 1,600 \quad J \approx 1,024$$

Aliasing Artifacts

- Nyquist Sampling theorem:
  - If the maximum freq of a signal is fmax, it should be sampled with a freq fs>=2max, or sampling interval T<=1/2fmax
  - If sampled at a lower freq. without pre-filtering, aliasing will occur
    - High freq. content fold over to low freq
  - Prefilter to lower fmax, and then sample

- If the number of samples in each projection (D) or the number of projection angles (M) are not sufficiently dense, the reconstructed image will have streak artifacts
  - Caused by aliasing
  - Practical detectors are area detectors and perform pre-filtering implicitly



From [Kak&Slaney] Fig. 5.1

Summary

- Parallel projection reconstruction
  - Backprojection summation
  - Fourier method (projection slice theorem)
  - Filtered backprojection
  - Convolution backprojection
  - Practical implementation: using finite samples
- Fan beam projection and reconstruction
  - Weighted backprojection
- Blurring due to non-ideal filters and detectors
  - Approximate the overall effect by a filter:
    - $h(l)=w(l)*s(l)$; $H(\rho)=W(\rho)\, S(\rho)$
  - Circularly symmetric functions and Hankel transform
    - Equivalent spatial domain filter h(r)=inverse Hankel {H(q)}
- Noise in measurement and reconstructed image
  - Factors influencing the SNR of reconstructed image
    - Number of angles (M), number of samples per angle (D), filter cut-off ρ_o
- Impact of number of projection angles and samples on reconstruction image quality
  - Nyquist sampling theorem
  - Streak artifacts

Reference

- Prince and Links, Medical Imaging Signals and Systems, Chap 6.
- A. C. Kak and M. Slaney, Principles of Computerized Tomographic Imaging. Originally published by IEEE, 1998. E-copy available at http://www.slaney.org/pct/
  - Chap 3 Contain detailed derivation of reconstruction algorithms both for parallel and fan beam projections. Have discussions both in continuous domain and implementation with sampled discrete signals.
  - Chap 5 discusses noise in measurement and reconstructed image.
  - Chap 5 also covers aliasing effect with more mathematical interpretations

Homework

- Reading:
  - Prince and Links, Medical Imaging Signals and Systems, Chap 6, Sec.6.3.4-6.5
- Note down all the corrections for Ch. 6 on your copy of the textbook based on the provided errata.
- Problems for Chap 6 of the text book:
  - P.6.9
  - P.6.10 (part e is not required)
  - P.6.13.
    - Hint: solution for part (a) should be

$$g(l,60)=\begin{cases}\sqrt{3}\mu(a/2+l) & -a/2\le l\le 0\\ \sqrt{3}\mu(a/2-l) & 0\le l\le a/2\\ 0 & otherwise\end{cases}$$

  - P.6.17
  - P.6.19
  - P.6.20

Computer Assignment

1. Learn how do ‘fanbeam’.’ifanbeam’ work; summarize their functionalities. Type ‘demos’ on the command line, then select ‘toolbox -> image processing -> transform -> reconstructing an image from projection data’. Alternatively, you can use ‘help’ for each particular function.
2. Write a MATLAB program that 1) generate a phantom image (you can use a standard phantom provided by MATLAB or construct your own), 2) produce equiangular fan beam projections; 3) reconstruct the phantom using filtered backprojection algorithm; Your program should allow the user to specify the number of fan beams, and the number of projections per fan beam, the angular spacing between the projections. Run your program with different number of projections for the same view angle, and with different view angles, and compare the quality. Use the same filter and interpolation algorithm for all the comparisons. Compare the reconstructed image quality obtained with different number of view angles and number of projections per view angle. Also, compare the image quality with those obtained with parallel projections for the same phantom image when the same total number of measurements are used (from your last assignment). You can use the “fanbeam()” and “ifanbeam()” functions in MATLAB.