ODR - Obyčejné dif. rovnice
Analytické metody

- definice - diferenciální, obyčejná
- $y(x) \in S$ 1. řádu $f(y', y, x) = 0$
  nejjednodušší $y' = g(y, x)$
  počáteční podmínky $y(0) = C$
- vyššího řádu, systémy
  Př: $y'' = 0$, $y = C_1 x + C_2$ - 2 počáteční podmínky
  2 okrajové podmínky
- existence řešení
  jednoznačnost řešení - bifurkace
- speciální případy
  - separace proměnných $y' = f(x) g(y)$
  $$\int \frac{dy}{g(y)} = \int f(x)\,dx + C$$
- lineární s konst. koef. - homogenní rovnice
  $$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = 0$$
  $$y = C e^{\lambda x}$$
  charakteristický polynom
  $$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$$
  kořeny $\lambda_1, \dots, \lambda_n$
  obecné řešení $y = \sum_{i=1}^{n} c_i e^{\lambda_i x}$

  násobné kořeny $c_i \sim A_i x + B_i$
- nehomogenní rovnice
  - partikulární řešení + obecné řešení homogenní
  - variace konstant $y = \sum_{i=1}^{n} C_i(x) e^{\lambda_i x}$
- teorie řízení - Laplaceova transformace
- numerické metody řešení
  - řady - Taylor
  - Picardova metoda $y' = f(x, y)$, $y(x_0) = y_0$
  $$y_{i+1}(x) = y_0 + \int_{x_0}^{x} f(s, y_i(s))\,ds$$

SYSTÉM ODR

$$\vec{Y}' = \vec{F}(x, \vec{Y})$$

- lineární

$$y_1' = \sum_{j=1}^{n} a_{1j}(x)\, y_j + b_1(x)$$

$$y_2' = \quad \dots$$

$$\vdots$$

$$y_n' = \sum_{j=1}^{n} a_{nj}(x)\, y_j + b_n(x)$$

$$\vec{Y}' = \mathbb{A}(x)\cdot\vec{Y} + \vec{B}(x)$$

- homogenní $\vec{B}(x) = 0$

$$\vec{Y}' = \mathbb{A}(x)\cdot\vec{Y}$$

$$\vec{Y} = \vec{C} e^{\lambda x}$$

$$\lambda \vec{C} e^{\lambda x} = \mathbb{A}\cdot\vec{C} e^{\lambda x}$$

$$\mathbb{A}\cdot\vec{C} = \lambda \vec{C}$$

$\det(\mathbb{A} - \lambda \mathbb{I}) = 0$

kořeny

- $\lambda_1, \lambda_2, \dots, \lambda_n$ – vlastní čísla matice $\mathbb{A}$

$\vec{C}_1, \vec{C}_2, \dots, \vec{C}_n$ – –"– vektory –"– $\mathbb{A}$

1. $\lambda_j \in R$, $\lambda_j \neq \lambda_k$, $j \neq k$, $j,k = 1, \dots, n$

obecné řešení

$$\vec{Y} = \sum_{j=1}^{n} d_j \vec{C}_j e^{\lambda_j x}, \quad d_j \in R$$

2. $\exists_j\ \lambda_j \notin R$, $\lambda_j \neq \lambda_k$, $j \neq k$, $j,k = 1, \dots, n$

$\lambda_j \notin R$, $\vec{C}_j$

$$\vec{C}_j e^{\lambda_j x} = (\mathrm{Re}\,\vec{C}_j + i\,\mathrm{Im}\,\vec{C}_j)\, e^{\mathrm{Re}\,\lambda_j x}\left(\cos(\mathrm{Im}\,\lambda_j x) + i \sin(\mathrm{Im}\,\lambda_j x)\right)$$

$$S_j^1 = \left(\operatorname{Re}\vec{C}_j \cdot \cos(\operatorname{Im}\lambda_j \cdot x) - \operatorname{Im}\vec{C}_j \cdot \sin(\operatorname{Im}\lambda_j \cdot x)\right) e^{\operatorname{Re}\lambda_j \cdot x} +$$

$$i\left(\operatorname{Im}\vec{C}_j \cdot \cos(\operatorname{Im}\lambda_j \cdot x) + \operatorname{Re}\vec{C}_j \cdot \sin(\operatorname{Im}\lambda_j \cdot x)\right) e^{\operatorname{Re}\lambda_j \cdot x} \quad S_j^2$$

$\lambda_j \notin \mathbb{R},\ \Rightarrow\ \overline{\lambda_j}$ je kořen char. polynomu

$(\vec{C}_j \qquad \overline{\vec{C}_j})$

$S_j^1,\ S_j^2 \in \mathbb{R}$

3. char. polynom má násobné kořeny

$\lambda_1 = \lambda_2$

a) $\vec{C}_1 \nparallel \vec{C}_2$ lin. nezávislé!

$$d_1 \vec{C}_1 e^{\lambda_1 x} + d_2 \vec{C}_2 e^{\lambda_2 x}$$

b) $\vec{C}_1 \parallel \vec{C}_2$ lin. závislé!

$$d_1 \vec{C}_1 e^{\lambda_1 x} + d_2 \vec{D}_1 e^{\lambda_1 x} = (d_1 x + d_2)\vec{C}_1 e^{\lambda_1 x}$$

$$(\mathbb{A} - \lambda_1 \mathbb{I})\vec{D}_1 = \vec{C}_1$$

ODR

Stabilita

Def:
ODR je stabilní $\Longleftrightarrow$ malé změně počátečních podmínek odpovídá malá změna řešení ODR (pro rostoucí x)

plot({y3(x,0.97),y3(x,1),

Př 1: $y' = -2y + e^{-x}$ $\quad y(0) = y_0 \sim 1$ – 0.97, 1, 1.03
$x \in (-3,3)$

Př 2: $y' = 2y - 3e^{-x}$ $\quad y(0) = y_0 \sim 1$ – 0.97, 1, 1.03
$x \in (-2,2)$

Věta: ODR $y' = f(x,y)$, $f \in C^1_{x,y}$
$\exists\ K, L$ $\quad K \leq \frac{\partial f}{\partial y} \leq L \quad \forall x, y,\ x \in (x_0, x_1)$
$y(x)$, $\tilde{y}(x)$ řešení ODR na intervalu $x \in (x_0, x_1)$
s poč. podmínkami $y(x_0) = y_0$, $\tilde{y}(x_0) = \tilde{y}_0$
$$\Rightarrow \quad |y_0 - \tilde{y}_0|\, e^{K(x-x_0)} \leq |y(x) - \tilde{y}(x)| \leq |y_0 - \tilde{y}_0|\, e^{L(x-x_0)}$$
$$\forall x \in (x_0, x_1)$$

Př 1: $\frac{\partial f}{\partial y} = -2$, $K = L = -2$
$$|y - \tilde{y}| = |y_0 - \tilde{y}_0|\, e^{-2(x-x_0)}$$

Př 2: $\frac{\partial f}{\partial y} = 2$, $K, L = 2$
$$|y - \tilde{y}| = |y_0 - \tilde{y}_0|\, e^{2(x-x_0)}$$

Př 3: $y' = -x^2 y^3 + \cos x$ $\qquad |y - \tilde{y}| \leq |y_0 - \tilde{y}_0|$
$\frac{\partial f}{\partial y} = -3x^2y^2 \leq 0^{=L} \Rightarrow$ stabilní

Věta: ODR, $y' = f(x,y)$, $f \in C^1_{x,y}$
$\forall x, y \quad \frac{\partial f}{\partial y} \leq 0 \Rightarrow$ ODR je stabilní

Runge-Kutta

- $y' = f(x) \qquad y = \int f(x)\,dx$

$$y(x+h) = y(x) + y'\left(x+\frac{h}{2}\right)\cdot h + \cancel{O(h^3)}$$

$$\approx y(x) + f\left(x+\frac{h}{2}\right)\cdot h$$

$$y_{n+1} = y_n + f\left(x_n+\frac{h}{2}\right)\cdot h + O(h^2)$$

- $y' = f(x,y)$

$$y(x+h) \approx y(x) + \overbrace{f\left(x+\frac{h}{2}, y\left(x+\frac{h}{2}\right)\right)}^{k_2} h$$

$$y\left(x+\frac{h}{2}\right) \approx y(x) + \overbrace{f(x, y(x))}^{k_1}\frac{h}{2}$$

$$k_1 = f(x_n, y(x_n))$$

$$k_2 = f\Big(\underbrace{x+\frac{h}{2}}_{x_{n+1/2}}, \underbrace{y(x)+k_1\frac{h}{2}}_{y_{n+1/2}}\Big)$$

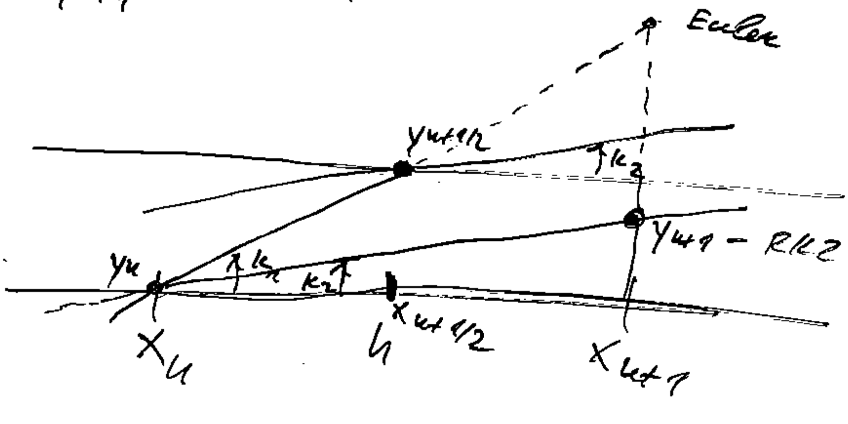
$$\underbrace{y(x_n+h)}_{y_{n+1}} \approx \underbrace{y(x)}_{y_n} + k_2\cdot h$$

$y' = f(x,y)$

RK2

$$\begin{array}{c|cc} 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

$O(h^2)$



RK metody

$$y' = f(x, y) \qquad y(x_0) = y_0$$

1. krok z x_0 do $x_1 = x_0 + h$ (Δx), $y_1 = y(x_1)$

$$k_1 = f(x_0, y_0)$$
$$k_2 = f(x_0 + c_2 h, y_0 + a_{21} k_1 h)$$
$$k_3 = f(x_0 + c_3 h, y_0 + h(a_{31} k_1 + a_{32} k_2))$$
$$\vdots$$
$$k_r = f\left(x_0 + c_r h, y_0 + h \sum_{j=1}^{r-1} a_{rj} k_j\right)$$
$$y_1 = y_0 + h \sum_{j=1}^{r} b_j k_j$$

$$\begin{array}{c|cccc} c_2 & a_{21} & & & \\ c_3 & a_{31} & a_{32} & & \\ \vdots & & & & \\ c_r & a_{r1} & a_{r2} & \dots & a_{r,r-1} \\ \hline & b_1 & b_2 & \dots\, b_{r-1} & b_r \end{array}$$

- obecná explicitní $\underbrace{\text{RK metoda}}_{r\text{-stupňová}}$
- i pro vektory
- nejjednodušší

(1) $r = 1$

$$k_1 = f(x_0, y_0)$$
$$y_1 = y_0 + h k_1 b_1$$
$$y(x_0 + h) = y(x_0) + b_1 h f(x_0, y_0)$$

$$L = y(x_0 + h) \approx y(x_0) + h y' + O(h^2)$$

(a) $L(h=0) = y(x_0)$ ✓

$P(h=0) = y(x_0)$

(b) $L'(h=0) = y'(x_0) = f(x_0, y_0)$

$P'(h=0) = b_1 f(x_0, y_0)$

$b_1 f = f$

$b_1 = 1$

$$k_1 = f(x_0, y_0)$$
$$\underline{y_1 = y_0 + h f(x_0, y_0)}$$

Eulerova metoda

② $r = 2$ $\quad k_1 = f(x_0, y_0)$ $\qquad y' = f(x, y)$

$$k_2 = f(x_0 + c_2 h,\ y_0 + a_{21} k_1 h)$$

$$y_1 = \underbrace{y_0 + h(b_1 k_1 + b_2 k_2)}_{P}$$

$$L = y(x_0 + h)$$

$y_0 = y(x_0)$

$y_1 = y(x_0 + h)$

ⓐ $L(h=0) = y_0$ ✓

$P(h=0) = y_0$

ⓑ $L'(h=0) = y'(x_0) = f(x_0, y_0)$

$$P'(h=0) = \frac{d}{dh} P \Big|_{h=0} = (b_1 k_1 + b_2 k_2)\Big|_{h=0} =$$

$$= b_1 f(x_0, y_0) + b_2 f(x_0, y_0) =$$

$$= f(b_1 + b_2)$$

$L' = P'$ $\quad f(b_1 + b_2) = f$

$$b_1 + b_2 = 1$$

ⓒ $$L''(h=0) = y''(x_0) = f'(x_0, y(x_0)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y'(x_0) = \overline{= f_x + f_y \cdot f}$$

$$P''(h=0) = \frac{d}{dh}\left[b_1 k_1 + b_2 k_2 + h \frac{d}{dh}(b_1 k_1 + b_2 k_2)\right] =$$

$$= 2 \frac{d}{dh}(b_1 k_1 + b_2 k_2) = 2 b_2 \frac{d}{dh} k_2 =$$

$$= 2 b_2 (f_x c_2 + f_y \cdot a_{21} k_1) = \underline{2 b_2 (f_x c_2 + f_y f a_{21})}$$

$\forall f$ $\qquad f_x + f_y \cdot f = 2 b_2 c_2 f_x + 2 b_2 a_{21} f_y f$

$$f_x (1 - 2 b_2 c_2) + f_y \cdot f (1 - 2 b_2 a_{21}) = 0$$

$$\boxed{\begin{aligned} 1 - 2 b_2 c_2 &= 0 \\ 1 - 2 b_2 a_{21} &= 0 \\ b_1 + b_2 &= 1 \end{aligned}}$$

$b_2 \neq 0$

$$\begin{array}{c|cc} 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

- vybereme $b_1 = 0 \Rightarrow b_2 = 1,\ c_2 = 1/2,\ a_{21} = 1/2$
- – " – $\quad b_1 = 1/2 \rightarrow b_2 = 1/2,\ c_2 = 1,\ a_{21} = 1$

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

3-krokové RK metoda   RK3

$k_1 = f(x_0, y_0)$

$k_2 = f(x_0 + c_2 h, y_0 + a_{21} k_1 h)$

$k_3 = f(x_0 + c_3 h, y_0 + h(a_{31} k_1 + a_{32} k_2))$

$y_1 = y_0 + h(b_1 k_1 + b_2 k_2 + b_3 k_3)$

- hledané konstanty $c_2, c_3, b_1, b_2, b_3, a_{21}, a_{31}, a_{32}$

$$\begin{array}{c|ccc} 0 & & & \\ c_2 & a_{21} & & \\ c_3 & a_{31} & a_{32} & \\ \hline & b_1 & b_2 & b_3 \end{array} \qquad \begin{array}{c|ccc} 0 & & & \\ 1/3 & 1/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

Heun - 3. řádu

- do 3. derivace
- rovnice 1. derivace $\to 1 - b_1 - b_2 - b_3 = 0$

2. derivace $1 - 2b_2 c_2 - 2 b_3 c_3 = 0$

$1 - 2b_2 a_{21} - 2b_3 a_{31} - 2b_3 a_{32} = 0$

3. derivace

$f_{xx} \to 1 - 3b_2 c_2^2 - 3b_3 c_3^2 = 0$

$f_{yy} \to 1 - 3b_2 a_{21}^2 - 3b_3 a_{31}^2 - 6b_3 a_{31} a_{32} - 3b_3 a_{32}^2 = 0$

$f_{xy} \to 2 - 6b_2 c_2 a_{21} - 6b_3 c_3 a_{31} - 6b_3 c_3 a_{32} = 0$

$f_y^2 \to 1 - 6b_3 a_{32} a_{21} = 0$

$f_x f_y \to 1 - 6b_3 a_{32} c_2 = 0$

ODR numerické řešení

- Eulerova metoda
- ODR $\quad y' = f(x,y)\ ,\ y(x_0) = y_0$
- krok h – malý

$$y(x+h) = y(x) + h\,y'(x) + O(h^2)$$
$$\doteq y(x) + h\cdot f(x, y(x))$$

$y_n = y(x_0 + n\cdot h)\ ,\ x_n = x_0 + n\cdot h$

$y_0 = y_0 \quad$ P.P.

$$y_{n+1} = y_n + h\cdot f(x_n, y_n)$$

cp ~liska/vyuka/odos/euler.m .

euler(x_0, y_0, x_1, h) $\qquad y' = -y$, $y(0) = 1$, $y = e^{-x}$

matlab

euler(0,1,40,0.1);

80	0.5	$h\lambda \in (-2,0)$
	1	$\lambda = -1$, $R = 1 + h\lambda$
	1.5	$h \in (0,2)$
	2	
	2.5	$\lvert 1 + h\lambda \rvert \le 1$
	2.1	$-2 \le h\lambda \le 0$
	1.9	$0 \le h \le 2$

• Eulerova metoda, stabilita

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$y' = f(x, y)$$

$y = 0$, $f(x, 0) = 0$

$$f(x, y) = f(x, 0) + \frac{\partial f}{\partial y} \cdot y + O(y^2)$$

$$\sim \underbrace{\frac{\partial f}{\partial y}}_{\sim J} \cdot y = J \cdot y$$

$$y_{n+1} = y_n + h J y_n$$

$$= \underbrace{(1 + hJ)}\, y_n$$

funkce stability $R(hJ) \sim R(z)$, $z \in \mathbb{C}$

$$y_{n+1} = R(h^n J^n)\, y_n$$

$$y_{n+m} = \prod \left(R(h^i J^i)\right)^m y_n = \prod \left(R(h^i J^i)\right) y_0$$

podmínka stability $|R(h^i J^i)| \leq 1$, $\forall i$

skalární případ

systém n rovnic J - $n \times n$ matice

λ_j, $\vec{v}_j$ - n vl. čísla, vl. vektory J

$$\vec{y}_n = \sum_{j=1}^{n} c_j \vec{v}_j$$

$$y_{n+1} = \sum c_j R(hJ \vec{v}_j) = \sum^{n} c_j R(h \lambda_j \vec{v}_j) =$$

$$= \sum_{j=1}^{n} c_j \cdot R(h\lambda_j)\vec{v}_j$$

$$y_{n+m} = \sum_{j=1}^{n} \left(R(h\lambda_j)\right)^m c_j \vec{v}_j$$

podmínka stability $|R(h\lambda_j)| \leq 1$

$\forall j = 1, \dots, n$

- Eulerova metoda

$z = hJ \leftarrow \frac{\partial f}{\partial y}$, $h\lambda_j \leftarrow \frac{\partial \vec{f}}{\partial \vec{y}}$

$$R(z) = 1 + z \quad , z \in \mathbb{C}$$

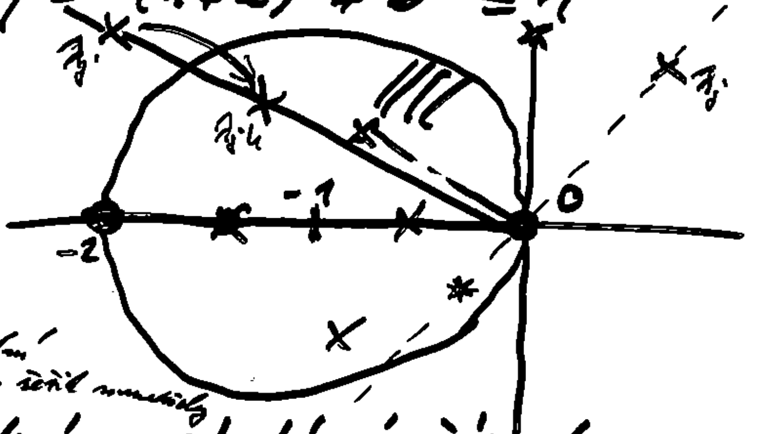
- obor stability

$$S_D = \{ z \in \mathbb{C}, |R(z)| \leq 1 \}$$

$$|1+z|^2 \leq 1 \qquad z = a + ib$$

$$|1+a+ib|^2 = (1+a)^2 + b^2 \leq 1$$

$z = \begin{cases} h f_y \text{ - skalární} \\ h\lambda_j \text{ - systém} \end{cases}$



$\left.\begin{array}{l} \mathrm{Re}\, f_y > 0 \\ \mathrm{Re}\, \lambda_j > 0 \end{array}\right\}$ ODR je nestabilní – nelze řešit numericky

- E.M. stabilní – skalární případ

$\boxed{hJ \in (-2, 0)}$ $\quad |1 + hJ| \leq 1$, $J = \frac{\partial f}{\partial y}$

1. $1 + hJ > 0 \qquad 1 + hJ \leq 1 \quad -1 < hJ < 0$
2. $1 + hJ < 0 \qquad -1 - hJ \leq 1 \quad -2 < hJ < -1$

- E.M. stabilní – systém rovnic

$\lambda_j < 0$

$$|1 + h\lambda_j| \le 1 \quad \forall j = 1, \dots, m$$

$$\Updownarrow$$

$z_j = h\lambda_j$

$\forall j \; z_j \in$ obor stability

$$\boxed{h > 0}$$

- $Re\,\lambda_j > 0$ → nestabilní systém ODR

$$\vec{y}\,' = \mathbb{A} \cdot \vec{y} \qquad \text{lin. systém}$$

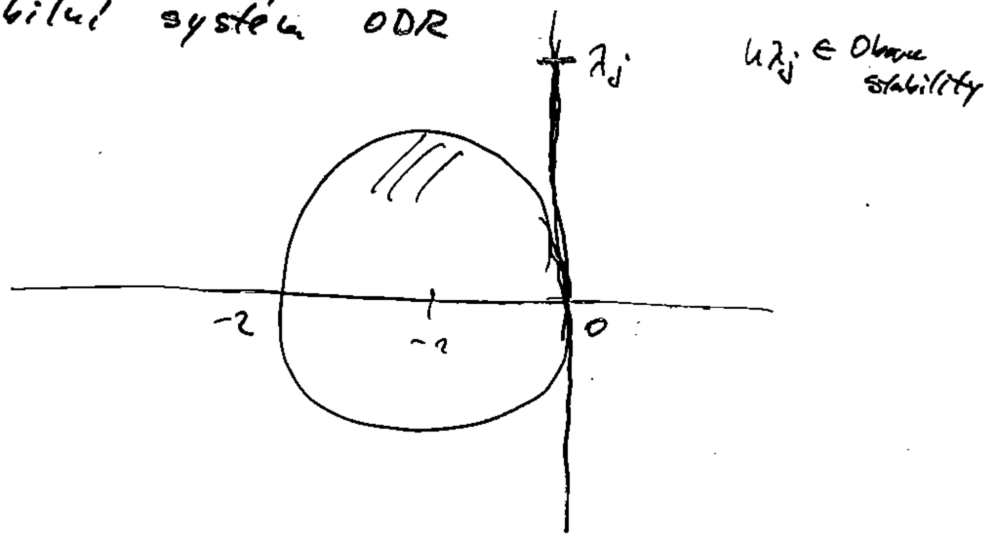
jakobián $\mathbb{J} = \mathbb{A}$

$$\sum_j d_j \vec{C}_j e^{\lambda_j x}$$

- $Re\,\lambda_j = 0$ – E.M. nelze použít

∃ RK metody, které lze použít

stabilní systém ODR



vliska/vyuka/odes/euler.m

```
function eu=euler(x0,y0,x1,h)
% Eulerova metoda pro reseni ODR
global n x y

N = round((x1 - x0)/h);
x = zeros(N,1);
y= zeros(N,1);

x(1) = x0;
y(1) = y0;
n = 1;
while x(n) <= x1
  y(n+1) = y(n) + h*f(x(n),y(n));
  x(n+1) = x(n) + h;
  n = n+1;
end;
plot(x,y);


function ff = f(x,y)
ff = -y;
```

euler(0, 10, 5, 0.1)

$y' = -y \;,\; y(x_0) = y_0$

$y = e^{\lambda x} \qquad y(0) = 10$

$\lambda = -1$

$y = c \cdot e^{-x} = y_0 e^{-x}$

$y' = -y \;,\; y(0) = 10$

$\underline{y = 10\, e^{-x}} \quad x \in (0, 5)$

$e_{max} = \max_n |y_n - y(x_n)|$

$e_{l_1} = \sum_n h \cdot |y_n - y(x_n)|$

$e_{l_2} = h \sqrt{\sum_n |y_n - y(x_n)|^2}$

$y' = -xy \;,\; y(0) = 1 \;,\; x \in (0, 5) \;,\; h = 0.1$

euler(0, 1, 5, 0.1)

50

100

RK2

$k_1 = f(x_n, y(x_n))$

$k_2 = f(x_n + h/2 \,,\, y(x_n) + k_1 \frac{h}{2})$

$y_{n+1} = y_n + k_2 h$

$$\begin{array}{c|cc} 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

RK3 Heun

$k_1 = f(x_n, y(x_n))$

$k_2 = f(x_n + h/3 \,,\, y_n + k_1 h/3)$

$k_3 = f(x_n + 2/3\, h \,,\, y_n + k_2 h \cdot 2/3)$

$y_{n+1} = y_n + \frac{1}{4} k_1 h + \frac{3}{4} k_3 h$

$$\begin{array}{c|ccc} 1/3 & 1/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

RK4

$k_1 = f(x_n, y(x_n))$

$k_2 = f(x_n + h/2 \,,\, y_n + k_1 h/2)$

$k_3 = f(x_n + h/2 \,,\, y_n + k_2 h/2)$

$k_4 = f(x_n + h \,,\, y_n + k_3 h)$

$y_{n+1} = \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_n$

$$\begin{array}{c|cccc} 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 2/6 & 2/6 & 1/6 \end{array}$$

euler. ... $\qquad O(h^4)$

RK4

$$
\begin{array}{c|cccc}
1/2 & 1/2 & & & \\
1/2 & 0 & 1/2 & & \\
1 & 0 & 0 & 1 & \\
\hline
 & 1/6 & 2/6 & 2/6 & 1/6
\end{array}
$$

$y' = -xy, \quad y(0) = 1$

$f(x,y) = -xy$

$f_y = -x$

Euler $\quad h f_y > -2$

$-hx > -2$

$h < \frac{2}{x}$

RK4: $\quad h < \frac{2.8}{x}$

$k_1 = f(x_n, y_n)$

$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{k_1 h}{2}\right)$

$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{k_2 h}{2}\right)$

$k_4 = f\left(x_n + h, y_n + k_3 h\right)$

$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$

$y' = -y, \quad y(0) = 10, \quad x \in (0, 10)$

řešení $\quad y = 10 e^{-x}$

h	0.4	0.2	0.1	0.01
RK4	10^{-3}	$6 \cdot 10^{-5}$	$3 \cdot 10^{-6}$	$3 \cdot 10^{-10}$ $O(h^4)$
RK3			$1.6 \cdot 10^{-4}$	$1.5 \cdot 10^{-7}$ $O(h^3)$
RK2			$6 \cdot 10^{-3}$	$6 \cdot 10^{-5}$ $O(h^2)$
Euler			0.19	0.018 $O(h)$

$O(h^4) \sim c \cdot h^4$

$h' = \frac{h}{10} \qquad c \cdot \frac{h^4}{10^4}$

ODR, Eulerova metoda pro systém

- systém $\vec{y}\,' = \vec{f}(x, \vec{y}) \qquad \vec{y}, \vec{f} \in R^m$

  předpoklad $\vec{f}(x, \vec{0}) = \vec{0}$, $|\vec{y}|$ malé

  $$\vec{f}(x, \vec{y}) = \vec{f}(x, \vec{0}) + \frac{\partial \vec{f}}{\partial \vec{y}} \cdot \vec{y} + O(|\vec{y}|^2)$$

  Jakobiho matice, Jakobián

  $$\sim \frac{\partial \vec{f}}{\partial \vec{y}} \cdot \vec{y} = \mathbb{J} \cdot \vec{y}$$

  $$\mathbb{J} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}, & \frac{\partial f_1}{\partial y_2}, & \dots, & \frac{\partial f_1}{\partial y_m} \\ \frac{\partial f_2}{\partial y_1}, & \frac{\partial f_2}{\partial y_2} & \dots & \\ & & \ddots & \\ \frac{\partial f_m}{\partial y_1} & \dots & & \frac{\partial f_m}{\partial y_m} \end{pmatrix}$$

- Eulerova metoda

  $$\vec{y}_{n+1} = \vec{y}_n + h\,\vec{f}(x_n, \vec{y}_n)$$
  $$\approx (\mathbb{I} + h\mathbb{J})\,\vec{y}_n$$

- funkce stability $\mathbb{R}(h\mathbb{J}) = \mathbb{I} + h\mathbb{J}$, $R(z) = 1 + z, z \in \mathbb{C}$
- vlastní čísla λ_j a vlastní vektory $\vec{v}_j$ Jakobiánu $\mathbb{J}$
- rozložíme

  $$\vec{y}_n = \sum_{j=1}^{m} c_j^n \vec{v}_j$$

  $$\vec{y}_{n+1} \approx (\mathbb{I} + h\mathbb{J}) \sum_{j=1}^{m} c_j \vec{v}_j = \sum_{j=1}^{m} c_j (\vec{v}_j + h\,\mathbb{J} \cdot \vec{v}_j) =$$
  $$= \sum_{j=1}^{m} c_j (\vec{v}_j + h\lambda_j \vec{v}_j) = \sum_{j=1}^{m} c_j (1 + h\lambda_j)\,\vec{v}_j$$
  $$= \sum_{j=1}^{m} c_j^n R(h\lambda_j)\,\vec{v}_j$$

- počáteční podmínka $\vec{y}_0 = \sum_{j=1}^{m} c_j^0 \vec{v}_j$
- obdobně

  $$\vec{y}_{n+1} = \sum_{j=1}^{m} \prod_i (1 + h^i \lambda_j^i)\, c_j^0 \vec{v}_j = \sum_{j=1}^{m} \prod_i R(h^i \lambda_j^i)\, c_j^0 \vec{v}_j$$

  stabilita $|R(h^i \lambda_j^i)| \le 1 \quad \forall i, \forall j$

Butcherova bariéra

Věta: Pro $p \geq 5$ neexistuje explicitní RK metoda řádu p, která by měla $s = p$ kroků.

Pro $p = 5$ dostaneme 17 rovnic pro 15 koeficientů c_i, b_i, a_{ij}. Tento systém už nemá řešení!

$s\backslash p$	1	2	3	4	5	6	7	8	9	10
rovnic	1	2	4	8	17	37	85	200	486	1205

Funkce stability

Dahlquistova testovací rovnice

$$y' = ay \quad , y_0 = 1 \, , \quad z = ah \quad , a \in \mathbb{C}$$

$$\begin{array}{c|cccc} c_2 & a_{21} & & & \\ c_3 & a_{31} & a_{32} & & \\ \vdots & \vdots & & & \\ c_s & a_{s1} & a_{s2} & \dots & a_{ss-1} \\ \hline & b_1 & b_2 & \dots \; b_{s-1} & b_s \end{array}$$

$$k_1 = f(x_0, y_0)$$
$$k_2 = f(x_0 + c_2 h, y_0 + a_{21} h \cdot k_1)$$
$$k_3 = f(x_0 + c_3 h, y_0 + h \cdot (a_{31} k_1 + a_{32} k_2)$$
$$\vdots$$
$$k_s$$
$$y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i$$

$$y_{n+1} = R(ah) y_n \approx R(z) y_n$$

$$R(z) = 1 + z \sum_j b_j + z^2 \sum_{j,k} b_j a_{jk} + z^3 \sum_{j,k,l} b_j a_{jk} a_{kl} + \dots$$

Def.: Obor stability RK metody je množina

$$S = \{ z \in \mathbb{C}, \; |R(z)| \leq 1 \}$$

- Krok metody h musí splňovat
  $$ha \in S$$
  potom je RK metoda stabilní

- rovnice $y' = f(x,y)$ $\qquad f_y = \frac{\partial f}{\partial y}$

  podmínka stability $h f_y(x) \in S \;\; \forall x$

- systém rovnic $Y' = F(x, Y)$

  podmínka stability $h \lambda_i \in S \;\; \forall i \quad \lambda_i(x)$ vlastní číslo $J = \frac{\partial F}{\partial Y}$ Jacobiho matice

$y' = ay \qquad a < 0$ ~liska/vyuka/odes/stab.mws

RK2 $\qquad f = ay \qquad J = \frac{\partial f}{\partial y} = a \quad , \; z = Jh = ah$

$$\text{RK2} \quad \begin{array}{c|cc} 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

$$\begin{aligned}
y_{n+1} &= y_n + k_2 h = \\
&= y_n + h\, f\!\left(x + \frac{h}{2}, y_n + k_1 \frac{h}{2}\right) = \\
&= y_n + ha\left(y_n + k_1 \frac{h}{2}\right) = \\
&= y_n + ha\left(y_n + \frac{h}{2} f(x_n, y_n)\right) = \\
&= y_n + ha\left(y_n + \frac{h}{2} a y_n\right) = \\
&= y_n\left(1 + ha + \frac{h^2a^2}{2}\right) = \\
&= y_n \underbrace{\left(1 + z + \frac{z^2}{2}\right)}_{R(z)}
\end{aligned}$$

$|R(z)|^2 \le 1$

$R(z)$ fce stability metody RK2 (obdélníková)

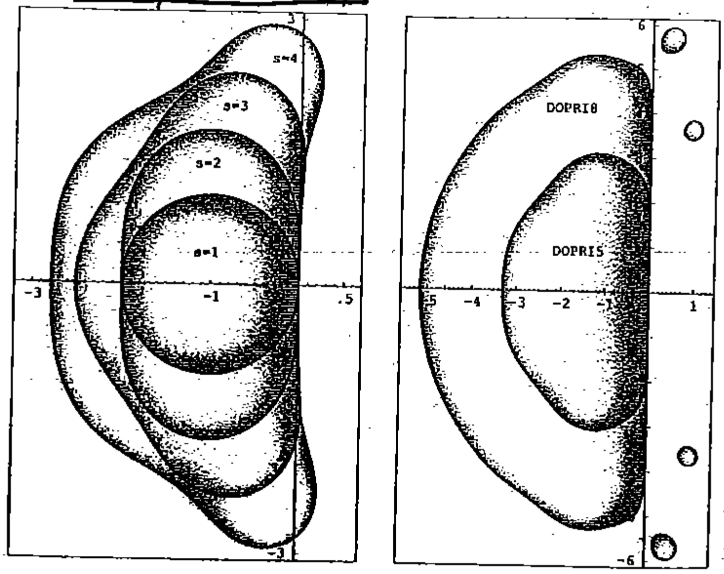
$R_3(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} \qquad$ RK3

$R_r(z) = \sum_{j=1}^{r} \frac{z^j}{j!} \quad$ – RK řádu přesnosti $O(h^r)$

$R_4(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} \qquad$ RK4

Obory stability

RKs



Dormand-Prince

Fig. 2.1. Stability domains for ERK methods of order $p = s$

Fig. 2.2. Stability domains for DOPRI methods

ODR

$y' = -xy, \quad y(0) = 5$

$f(x,y) = -xy$

$f_y = -x$

stabilní $\Leftrightarrow$ $hf_y \in$ oboru stability

$hf_y \in \mathbb{R}$

$0 \geq hf_y \geq -c$

$0 \geq -hx \geq -c$

$$\boxed{h \leq \frac{c}{x}}$$

metoda	c
Euler	2
RK2	2
RK3	2.5
RK4	2.8
DOPRI5	3.3
DOPRI8	5.1

Řád RK metody

Věta: RK metoda je p-tého řádu přesnosti

$$\Leftrightarrow \quad R(z) = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^p}{p!} + O(z^{p+1})$$

- pro explicitní s-krokovou RK metodu řádu přesnosti $p = s$ (tj. $p = s \leq 4$) platí

$$R(z) = 1 + z + \dots + \frac{z^p}{p!}$$

Embeded RK metody

- tabulka

$$\begin{array}{c|ccccc}
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & \vdots & & & \\
c_s & a_{s1} & a_{s2} & \dots & a_{s\,s-1} & \\
\hline
 & b_1 & b_2 & \dots & b_{s-1} & b_s \\
\hline
 & \hat{b}_1 & \hat{b}_2 & \dots & \hat{b}_{s-1} & \hat{b}_s
\end{array}$$

$$k_j = f(x + c_j h, y_n + h \sum a_{ji} k_i$$

$$y_{n+1} = y_n + h \sum_{j=1}^{s} b_j k_j \qquad RKp \text{, řádu } p \quad O(h^p)$$

$$\hat{y}_{n+1} = y_n + h \sum_{j=1}^{s} \hat{b}_j k_j \qquad RK\hat{p} \text{, řádu } \hat{p} \quad O(h^{\hat{p}})$$

$$\hat{p} = p+1 \vee \hat{p} = p-1$$

$$C \cdot h^{\hat{p}} < \varepsilon$$

$\| y_{n+1} - \hat{y}_{n+1} \|$ dává odhad přesnosti y_{n+1}

- slouží k adaptivnímu určování kroku h tak aby výsledné y_{n+1} mělo požadovanou přesnost
- metoda $RKp(\hat{p})$
  - nejčastěji používaná $RK4(5) \Leftrightarrow RKF45$

Runge–Kutta–Felberg

ODR s okrajovými podmínkami

$y'' = f(x, y, y')$ počáteční problém $y(x_0) = y_0$

$y'(x_0) = y_0'$

$\Updownarrow$

systému $y_1 = y$

$y_2 = y'$

$y_2' = f(x, y_1, y_2)$ $\quad y_1(x_0) = y_0$

$y_1' = y_2$ $\quad y_2(x_0) = y_0'$

okrajový problém $x \in (x_0, x_1)$

1. $y'' = f(x, y, y')$ $\quad y(x_0) = y_0, \; y(x_1) = \overline{y}_1$

2. $y_1' = y_2$
   $y_2' = f(x, y_1, y_2)$ $\quad y_1(x_0) = y_0, \; y_1(x_1) = \overline{y}_1$

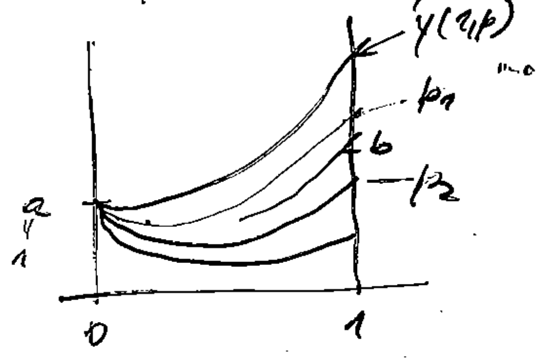
3. $y_1' = f_1(x, y_1, y_2)$ $\quad y_1(x_0) = y_0 \quad y_2(x_1) = \overline{y}_2$
   $y_2' = f_2(x, y_1, y_2)$

Př: $y'' = e^y$ $\quad y(0) = a, \; y(1) = b$

Metoda střelby

$y'' = e^y$ $\quad \begin{matrix} y(0) = a \\ y(1) = b \end{matrix}, \; y'(0) = p$

$y(1, p) = b$



monotonní

$p = \frac{p_1 + p_2}{2}$

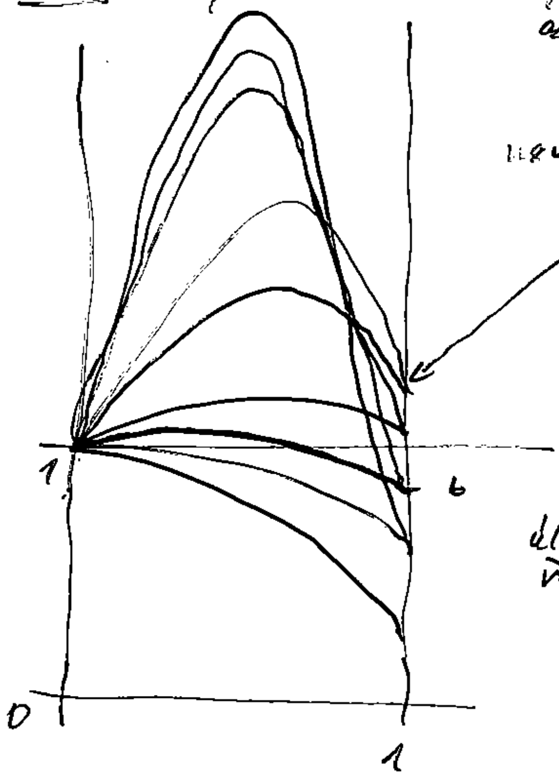
Př 2: $y'' = -e^y$ $\quad y(0)=a, \; y(1)=b$

$a = 1$

$y'(0) = p$

není monotónní

$y(1,p)$

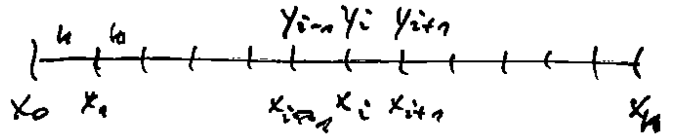


2 řešení

úloha nemá jednoznačné řešení

Metoda konečných diferencí

$$y'' = f(x, y, y'), \quad y(x_0) = a, \quad y(x_n) = b, \quad x \in (x_0, x_n)$$



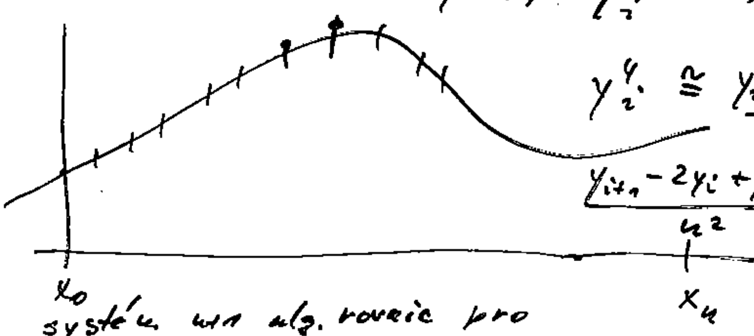
$$h = x_{i+1} - x_i = \frac{x_n - x_0}{n}$$

$$x_i = x_0 + i \cdot h$$

$y(x), \; x \in (x_0, x_n) \sim y_i, \; i = 0, \dots, n$

$y_i \approx y(x_i)$ $\quad y_i' \cong \dfrac{y_{i+1} - y_{i-1}}{2h}$

$$y_i'' \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$



$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$$

$i = 1, \dots, n-1$

systém n+1 alg. rovnic pro n+1 neznámých y_i, $i = 0, \dots, n$

$y_0 = a, \; y_n = b$