Strong oscillation (pumping) ①

$$Z(t) = 2Z_0 \cos\omega_0 t$$

2 normal modes

$$\mathcal{L}X(t) = \left\{\frac{d^2}{dt^2} + 2\Gamma_1\frac{d}{dt} + \omega_1^2\right\}X(t) = \lambda Y(t)Z(t)$$

$$\mathcal{K}Y(t) = \left\{\frac{d^2}{dt^2} + 2\Gamma_2\frac{d}{dt} + \omega_2^2\right\}Y(t) = \mu X(t)Z(t)$$

$\lambda\mu$ = Real > 0 assume $|\omega_1| \leq |\omega_2|$

Fourier transform

$$X(t), Y(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} X(\omega), Y(\omega)$$

$$[\omega^2 - \omega_1^2 + 2i\Gamma_1\omega]X(\omega) + \lambda Z_0[Y(\omega-\omega_0) + Y(\omega+\omega_0)] = 0$$

$$[\omega^2 - \omega_2^2 + 2i\Gamma_2\omega]Y(\omega) + \mu Z_0[X(\omega-\omega_0) + X(\omega+\omega_0)] = 0$$

Let $\omega_0 \simeq \omega_1 + \omega_2$ $\mathrm{Re}\,\omega \simeq \omega_1$ $(\omega = x + iy)$

$X(\omega)$ < $Y(\omega-\omega_0)$, $Y(\omega+\omega_0)$

$Y(\omega-\omega_0)$ → $X(\omega-2\omega_0)$ – out of resonance; $X(\omega)$

$Y(\omega+\omega_0)$ → $X(\omega)$; $X(\omega+2\omega_0)$ – out of resonance

$$\begin{vmatrix} [\omega^2-\omega_1^2+2i\Gamma_1\omega] & \lambda Z_0 & \lambda Z_0 \\ \mu Z_0 & [(\omega-\omega_0)^2-\omega_2^2+2i\Gamma_2(\omega-\omega_0)] & 0 \\ \mu Z_0 & 0 & [(\omega+\omega_0)^2-\omega_2^2+2i\Gamma_2(\omega+\omega_0)] \end{vmatrix} = 0$$

2 cases 2

a) $|\omega_1| \cancel{\simeq} |\omega_2|$

$Y(\omega+\omega_0)$ out of resonance $\omega+\omega_0 \simeq \omega_2+2\omega_1$

2×2 matrix

dispersion eq.

$$\left[\omega-\omega_1+i\Gamma_1\right]\left[\omega-\omega_0+\omega_2+i\Gamma_2\right]+\frac{\lambda\mu Z_0^2}{4\omega_1\omega_2}=0$$

frequency mismatch $\Delta = \omega_0-\omega_1-\omega_2$

normalized incident power $K=\lambda\mu Z_0^2$

$\omega = x+iy$

$$(y+\Gamma_1)(y+\Gamma_2)\left\{1+\frac{\Delta^2}{(2y+\Gamma_1+\Gamma_2)^2}\right\}=\frac{K}{4\omega_1\omega_2}$$

lowest threshold for $\Delta=0$

$$K_T = 4\omega_1\omega_2\Gamma_1\Gamma_2 = K_m \qquad K_T(\Delta)=K_m\left\{1+\frac{\Delta^2}{(\Gamma_1+\Gamma_2)^2}\right\}$$

at threshold $x=\omega_1+\frac{\Gamma_1}{\Gamma_1+\Gamma_2}\Delta$

for $K>>K_T$ $\quad y=\frac{1}{2}\sqrt{\frac{K}{\omega_1\omega_2}-\Delta^2} \quad x=\omega_1+\frac{\Delta}{2}$

b) $|\omega_1|<<|\omega_2|$ $\quad Y(\omega+\omega_0)$ also in resonance

two branches

a) $\omega = iy$ aperiodic instability only for $\omega_0<\omega_2$

b) $\omega = x+iy$ instability only for $\omega_0>\omega_2$

if $\omega_1^2>>\Gamma_1\Gamma_2 \Rightarrow x\simeq\omega_1$

$\omega_1^2<<\Gamma_1\Gamma_2 \Rightarrow x = f(\Gamma_1,\Gamma_2)$