Rutherford scattering

- binary elastic collision of 2 charged particles
- we want to find the scattering angle

Analytically solvable problem of motion in a central force field with potential $U(r)$

Total energy in laboratory coordinate system

$$E = 1/2\, m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + U(|\vec{r}_1 - \vec{r}_2|)$$

Energy in center of mass coordinate system (těžišťová soustava)

$$\vec{R}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \qquad \vec{V}_c = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} \qquad M_c = m_1 + m_2$$

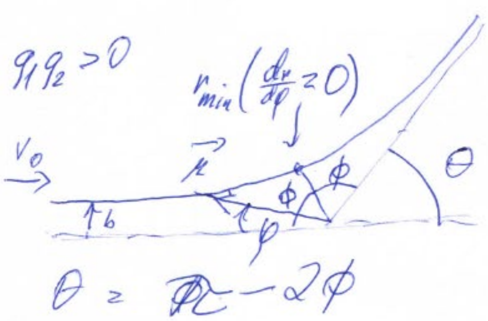
$$\vec{v}_1 = \vec{V}_c - \frac{m_2}{m_1+m_2}\underbrace{(\vec{v}_2 - \vec{v}_1)}_{\vec{v}} \qquad \mu = \frac{m_1 m_2}{m_1+m_2} \qquad \vec{r} = \vec{r}_2 - \vec{r}_1$$

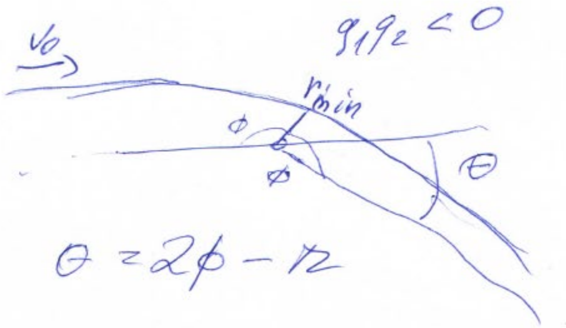
$$E = 1/2\, \mu \vec{v}^2 + 1/2\, M_c V_c^2 + U(r)$$

Angular momentum (moment hybnosti)

$$\vec{M} = m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2 = M_c \vec{R}_c \times \vec{V}_c + \mu \vec{r} \times \vec{v}$$

Problem may be transformed to scattering of particle of mass μ on a static scattering center





Energy conservation

$$\frac{1}{2}\mu v_0^2 = \frac{1}{2}\mu\left[\dot{r}^2 + r^2\dot{\varphi}^2\right] + U(r)$$

Conservation of angular momentum

$$\mu b v_0 = \mu r v_\varphi = \mu r^2 \dot{\varphi}$$

$$U = \frac{C}{r} \qquad C = \frac{q_1 q_2}{4\pi\varepsilon_0}$$

$$\dot{\varphi} = \frac{b v_0}{r^2} \qquad \dot{r} = \pm v_0 \sqrt{1 - \frac{b^2}{r^2} - \frac{2C}{\mu v_0^2 r}}$$

Then

$$\frac{d\varphi}{dr} = \frac{\dot{\varphi}}{\dot{r}} = \pm \frac{b}{r^2\sqrt{1 - \frac{b^2}{r^2} - \frac{2C}{\mu v_0^2 r}}}$$

In the beginning *r* decreases (-), then it grows (+)

$$\theta = \pi - 2\Phi = \pi - 2\int_{r_{min}}^{\infty} \frac{b/r^2\, dr}{\sqrt{1 - \frac{b^2}{r^2} - \frac{2b_0}{r}}}$$

b_0 - Landau length – $b_0 = C/(\mu v_0^2)$

Substitution

$$s = \frac{b}{r} + \frac{b_0}{b} \qquad ds = -\frac{b\, dr}{r^2} \qquad s_0^2 = 1 + \frac{b_0^2}{b^2}$$

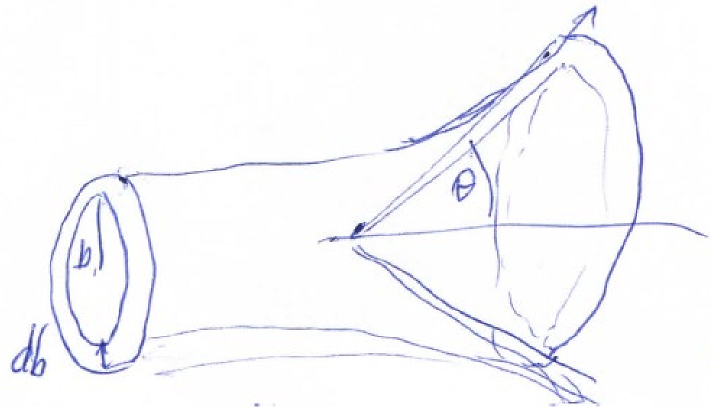
$$\theta = \pi - 2\int_{b_0/b}^{s_0} \frac{ds}{\sqrt{s_0^2 - s^2}} = \pi + 2 \left. \arccos\frac{s}{s_0} \right|_{b_0/b}^{b_0}$$

$$\theta = \pi - 2\arccos\frac{b_0/b}{\sqrt{1 + b_0^2/b^2}} = 2\,\mathrm{arctg}\,\frac{b_0}{b}$$

For $b = b_0$ angle $\theta = \pi/2$ (90°)

Differential cross section

$d\sigma$ - number of particles that scatter on 1 target particle to spherical angle $d\Omega$ during time unit divided by flux density of incident particles



$$\theta = f(b)$$
$$d\Omega = 2\pi \sin\theta \, d\theta$$
$$d\sigma = 2\pi b \, db$$

$$\left|\frac{d\sigma}{d\Omega}\right| = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$b = b_0 / tg\frac{\theta}{2} \qquad \frac{db}{d\theta} = \frac{b_0}{tg^2\frac{\theta}{2}} \cdot \frac{1}{2\cos^2\frac{\theta}{2}}$$

$$\left|\frac{d\sigma}{d\Omega}\right| = \frac{b_0^2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \cdot tg\frac{\theta}{2}\, tg^2\frac{\theta}{2}\, 2\cos^2\frac{\theta}{2}} = \frac{b_0^2}{4\sin^4\frac{\theta}{2}}$$

For collision e^- i^+

$$\left|\frac{d\sigma}{d\Omega}\right| = \left(\frac{Ze^2}{8\pi\varepsilon_0 \mu v_0^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} \cong \left(\frac{Ze^2}{8\pi\varepsilon_0 m_e v_0^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

Big for small θ, small for high velocity v_0

Total cross section

$$\sigma = \int_0^\pi \left|\frac{d\sigma}{d\Omega}\right| d\Omega = 2\pi \int_0^\pi \left|\frac{d\sigma}{d\Omega}\right| \sin\theta \, d\theta = \left[\theta = 2\,arctg\left(\frac{b_0}{b}\right)\right]$$

$$\sigma = 2\pi \int_0^\infty b \, db = \pi \lambda_D^2$$

Total cross section diverges, but Coulomb potential not valid for $b > \lambda_{De}$, thus $\infty \rightarrow \lambda_{De}$

However, we need cross section for some physical quantity,
most often momentum (σ_H) (or energy)

$$\sigma_H = 2\pi \int_{\theta_{min}} (1-\cos\theta)\left|\frac{d\sigma}{d\Omega}\right| \sin\theta\, d\theta =$$

$$= \frac{\pi b_0^2}{2} \int_{\theta_{min}}^{\pi} (1-\cos\theta)\frac{\sin\theta\, d\theta}{\sin^4\frac{\theta}{2}} = \left| \begin{array}{l} \psi = 1-\cos\theta \\ \sin\theta\, d\theta = d\psi \\ \psi = 2\sin^2\frac{\theta}{2} \end{array} \right.$$

$$= 2\pi b_0^2 \int_{2\sin^2\frac{\theta_{min}}{2}}^{2} \frac{d\psi}{\psi} = 2\pi b_0^2 \ln\frac{2}{2\sin^2\frac{\theta_{min}}{2}} =$$

$$= 2\pi b_0^2 \ln\left(1+\frac{1}{tg^2\frac{\theta_{min}}{2}}\right) = 2\pi b_0^2 \ln\left(1+\frac{\lambda_D^2}{b_0^2}\right) =$$

$$= 4\pi b_0^2 \ln \underbrace{\sqrt{1+\frac{\lambda_D^2}{b_0^2}}}_{\Lambda}$$

$$\sigma_{Hei} = \frac{Z^2 e^4}{4\pi\varepsilon_0^2 m_e^2 v^4} \ln\Lambda$$

$\ln \Lambda$ - Coulomb logarithm

electron mean free path

$$l_{mfp}^{ei} = (\sigma_{Hei}\, n_i)^{-1} \sim v^4$$

collision frequency

$$\nu_{ei} = n_i \langle \sigma v \rangle_{f(v)} \sim \frac{n_i}{T_e^{3/2}}$$

For calculation of cross section for energy exchange one needs to transform scattering results from center of mass coordinates to laboratory coordinates
Quantum description more accurate