Liouville equation (theorem)

Density of systems (× particle density = Klimontovich equation)
system of 1. particle (6-dimensional space)

$$N(\vec{x}_1, \vec{p}_1, t) = \delta[\vec{x}_1 - \vec{X}_1(t)]\,\delta[\vec{p}_1 - \vec{P}_1(t)]$$

System of N_0 particles ($6N_0$-dimensional space)

$$N(\vec{x}_1, \vec{p}_1, ..., \vec{x}_{N_0}, \vec{p}_{N_0}, t) = \prod_{i=1}^{N_0} \delta[\vec{x}_i - \vec{X}_i(t)]\,\delta[\vec{p}_i - \vec{P}_i(t)]$$

Normalization (1 system) $\int N(\vec{x}_1, \vec{p}_1, ..., \vec{x}_{N_0}, \vec{p}_{N_0}, t)\, \mathrm{d}\vec{x}_1\, \mathrm{d}\vec{p}_1 ... d\vec{x}_{N_0}\, \mathrm{d}\vec{p}_{N_0} = 1$

We express $\partial N / \partial t$ and use $\frac{\partial}{\partial t}\delta[\vec{x}_i - \vec{X}_i(t)] = -\frac{\mathrm{d}\vec{X}_i}{\mathrm{d}t}\,\nabla_{x_i}\delta[\vec{x}_i - \vec{X}_i(t)]$

Then one obtains $$\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \frac{\mathrm{d}\vec{X}_i}{\mathrm{d}t}\,\nabla_{\vec{x}_i} N + \sum_{i=1}^{N_0} \frac{\mathrm{d}\vec{P}_i}{\mathrm{d}t}\,\nabla_{\vec{p}_i} N = 0$$

and after substitution Liouville equation is obtained

$$\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \frac{\vec{\mathrm{p}}_i}{m_i}\nabla_{\vec{x}_i} N + \sum_{i=1}^{N_0} q_i \left[\vec{E}^m(\vec{x}_i, t) + \frac{\vec{\mathrm{p}}_i}{m_i} \times \vec{B}^m(\vec{x}_i, t)\right]\nabla_{\vec{p}_i} N = 0$$

Suggested reading: Nicholson chap. 4 and 5

Left side is total derivative with respect to the system trajectory $\frac{DN}{Dt} = 0$

Conservation law of number of the systems

$$\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \nabla_{\vec{x}_i} \left(\frac{\vec{p}_i}{m_i} N\right) + \sum_{i=1}^{N_0} \nabla_{\vec{p}_i} \left(\vec{F}\, N\right) = 0$$

Ensemble of macroscopically identical systems

Probability density

$$\tilde{f}_{N_0}(\vec{x}_1, \vec{p}_1, \ldots, \vec{x}_{N_0}, \vec{p}_{N_0})\, \mathrm{d}\,\vec{x}_1\, \mathrm{d}\,\vec{p}_1 \ldots \mathrm{d}\,\vec{x}_{N_0}\, \mathrm{d}\,\vec{p}_{N_0} = \frac{N_\Delta}{N_A}$$

ratio of number of systems in phase space element to all systems in ensemble

Integral = 1   systems in ensemble are neither born not get lost

$$\frac{\partial \tilde{f}_{N_0}}{\partial t} + \sum_{i=1}^{N_0} \nabla_{\vec{x}_i} (\dot{\vec{X}}_i\ \tilde{f}_{N_0}) + \sum_{i=1}^{N_0} \nabla_{\vec{p}_i} \left(\dot{\vec{P}}_i\ \tilde{f}_{N_0}\right) = 0$$

Density of systems does not change along system trajectory in phase space $D\,\tilde{f}_{N_0} / D t = 0$

N_A systems in ensemble
$$\tilde{f}_{N_0} = \frac{1}{N_A} \sum_{j=1}^{N_\Delta} \prod_{i=1}^{N_0} \delta[\vec{x}_i - \vec{X}_i(t)]\, \delta[\vec{p}_i - \vec{P}_i(t)]$$

Liouville´s equation
$$\frac{\partial \tilde{f}_{N_0}}{\partial t} + \sum_{i=1}^{N_0} \vec{\mathrm{v}}_i\, \nabla_{\vec{x}_i}\, \tilde{f}_{N_0} + \sum_{i=1}^{N_0} \vec{F}_i\, \nabla_{\vec{p}_i}\, \tilde{f}_{N_0} = 0$$

V is normalization volume
$$\int \tilde{f}_{N_0}\, \mathrm{d}\,\vec{p}_1\, \mathrm{d}\,\vec{p}_2 \ldots \mathrm{d}\,\vec{p}_{N_0} = V^{-N_0} \Rightarrow f_{N_0} = V^{N_0}\, \tilde{f}_{N_0}$$

***k*-particle distribution function**
$$f_k(\vec{x}_1, \vec{p}_1, \ldots, \vec{x}_k, \vec{p}_k) = V^{k-N_0} \int \mathrm{d}\,\vec{x}_{k+1}\, \mathrm{d}\,\vec{p}_{k+1}\, \mathrm{d}\,\vec{x}_{k+2}\, \mathrm{d}\,\vec{p}_{k+2} \ldots \mathrm{d}\,\vec{x}_{N_0}\, \mathrm{d}\,\vec{p}_{N_0}\, f_{N_0}$$

boundaries
$$f_{N_0} \to 0 \qquad x_i \vee y_i \vee z_i \to \pm\infty \qquad f_{N_0} \to 0 \qquad p_{xi} \vee p_{yi} \vee p_{zi} \to \pm\infty$$

distribution function is symmetric to interchange of identic particles
$$f_k(\ldots, \vec{x}_m, \vec{p}_m, \ldots, \vec{x}_n, \vec{p}_n, \ldots.) = f_k(\ldots, \vec{x}_n, \vec{p}_n, \ldots, \vec{x}_m, \vec{p}_m, \ldots.)$$

Let's assume 1 kind of particles for simplicity only Coulomb interactions

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{P}_i = \sum_{j=1}^{N_0} \vec{F}_{ij} \qquad \vec{F}_{ij} = \frac{1}{4\pi\,\varepsilon_0} \frac{q_s^2}{\left|\vec{x}_i - \vec{x}_j\right|^3}\left(\vec{x}_i - \vec{x}_j\right)$$

We derive equation for f_{N_0-1} from eq. for f_{N_0} by integration $\int \mathrm{d}\vec{x}_{N_0}\,\mathrm{d}\,\vec{p}_{N_0}$

$$\int \mathrm{d}\vec{x}_{N_0}\,\mathrm{d}\,\vec{p}_{N_0}\,\frac{\partial f_{N_0}}{\partial t} = V\frac{\partial}{\partial t}\,f_{N_0-1}$$

$$\int \mathrm{d}\vec{x}_{N_0}\,\mathrm{d}\,\vec{p}_{N_0} \sum_{i=1}^{N_0} \vec{\mathrm{v}}_i\,\nabla_{\vec{x}_i} f_{N_0} = V\sum_{i=1}^{N_0-1} \vec{\mathrm{v}}_i\,\nabla_{\vec{x}_i} f_{N_0-1} + \int \mathrm{d}\vec{x}_{N_0}\,\mathrm{d}\,\vec{p}_{N_0}\left(\mathrm{v}_{xN_0}\frac{\partial}{\partial x_{N_0}} + \mathrm{v}_{yN_0}\frac{\partial}{\partial y_{N_0}} + \mathrm{v}_{zN_0}\frac{\partial}{\partial z_{N_0}}\right) f_{N_0}$$

second integral = 0 (first term $\int \mathrm{d}\vec{p}_{N_0}\,\mathrm{d}\,y_{N_0}\,\mathrm{d}\,z_{N_0}\,\mathrm{v}_{xN_0}\,f_{N_0}\Big|_{x_{N_0}=-\infty}^{x_{N_0}=+\infty} = 0$)

$$\int \mathrm{d}\vec{x}_{N_0}\,\mathrm{d}\,\vec{p}_{N_0} \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} \vec{F}_{ij}\nabla_{\vec{p}_i} f_{N_0} = \sum_{i=1}^{N_0-1}\sum_{j=1}^{N_0-1} \vec{g}_{ij} + \sum_{i=1}^{N_0-1} \vec{g}_{iN_0} + \sum_{j=1}^{N_0-1} \vec{g}_{N_0 j} + \vec{g}_{N_0 N_0} = V\sum_{i=1}^{N_0-1}\sum_{j=1}^{N_0-1} \vec{F}_{ij}\nabla_{\vec{p}_i} f_{N_0-1} +$$

$$+\sum_{i=1}^{N_0-1}\int \mathrm{d}\vec{x}_{N_0}\,\mathrm{d}\,\vec{p}_{N_0}\vec{F}_{iN_0}\nabla_{\vec{p}_i} f_{N_0} + \sum_{j=1}^{N_0-1}\int \mathrm{d}\vec{x}_{N_0}\,\mathrm{d}\,p_{yN_0}\,\mathrm{d}\,p_{zN_0} F_{xN_0 j}\, f_{N_0}\Big|_{p_{xN_0}=-\infty}^{p_{xN_0}=+\infty} + 0$$

Equation for f_{N_0-1}

$$\frac{\partial}{\partial t} f_{N_0-1} + \sum_{i=1}^{N_0-1} \vec{\mathrm{v}}_i\,\nabla_{\vec{x}_i} f_{N_0-1} + \sum_{i=1}^{N_0-1}\sum_{j=1}^{N_0-1} \vec{F}_{ij}\nabla_{\vec{p}_i} f_{N_0-1} = -\frac{1}{V}\sum_{i=1}^{N_0-1}\int \mathrm{d}\vec{x}_{N_0}\,\mathrm{d}\,\vec{p}_{N_0}\vec{F}_{iN_0}\nabla_{\vec{p}_i} f_{N_0}$$

We derive eq. for f_{N_0-2} from eq. for f_{N_0-1} by integration $\int \mathrm{d}\vec{x}_{N_0-1}\,\mathrm{d}\,\vec{p}_{N_0-1}$

$$\sum_{i=1}^{N_0-1}\int \mathrm{d}\vec{x}_{N_0-1}\,\mathrm{d}\,\vec{p}_{N_0-1}\int \mathrm{d}\vec{x}_{N_0}\,\mathrm{d}\,\vec{p}_{N_0}\vec{F}_{iN_0}\nabla_{\vec{p}_i} f_{N_0} = \sum_{i=1}^{N_0-2}\int \mathrm{d}\vec{x}_{N_0-1}\,\mathrm{d}\,\vec{p}_{N_0-1}\vec{F}_{iN_0-1}\nabla_{\vec{p}_i}\int \mathrm{d}\vec{x}_{N_0}\,\mathrm{d}\,\vec{p}_{N_0} f_{N_0} = V\sum_{i=1}^{N_0-2}\int \mathrm{d}\vec{x}_{N_0-1}\,\mathrm{d}\,\vec{p}_{N_0-1}\vec{F}_{iN_0-1}\nabla_{\vec{p}_i} f_{N_0-1}$$

Equation for f_{N_0-2}

$$\frac{\partial}{\partial t} f_{N_0-2} + \sum_{i=1}^{N_0-2}\vec{\mathrm{v}}_i\nabla_{\vec{x}_i} f_{N_0-2} + \sum_{i=1}^{N_0-2}\sum_{j=1}^{N_0-2}\vec{F}_{ij}\nabla_{\vec{p}_i} f_{N_0-2} = -\frac{2}{V}\sum_{i=1}^{N_0-2}\int \mathrm{d}\vec{x}_{N_0-1}\,\mathrm{d}\,\vec{p}_{N_0-1}\vec{F}_{iN_0-1}\nabla_{\vec{p}_i} f_{N_0-1}$$

Equation for f_k

$$\frac{\partial}{\partial t} f_k + \sum_{i=1}^{k}\vec{\mathrm{v}}_i\nabla_{\vec{x}_i} f_k + \sum_{i=1}^{k}\sum_{j=1}^{k}\vec{F}_{ij}\nabla_{\vec{p}_i} f_k = -\frac{N_0-k}{V}\sum_{i=1}^{k}\int \mathrm{d}\vec{x}_{k+1}\,\mathrm{d}\,\vec{p}_{k+1}\vec{F}_{ik+1}\nabla_{\vec{p}_i} f_{k+1}$$

BBGKY *(Bogoljubov, Born, Green, Kirkwood, Yvon)* **hierarchy**

Equivalent to Liouville theorem (includes trajectories of ∀ particles), but is can be truncated, we are interested in $k << N_0$, then $\frac{N_0-k}{V} \simeq n_0$
(n_0 is particle density)

for one-particle distribution $\frac{\partial}{\partial t} f_1 + \vec{\mathrm{v}}_1\nabla_{\vec{x}_1} f_1 + n_0\int \mathrm{d}\vec{x}_2\,\mathrm{d}\,\vec{p}_2\vec{F}_{12}\nabla_{\vec{p}_1} f_2 = 0$

normalization condition $\int f_1(\vec{p}_1)\,\mathrm{d}\vec{p}_1 = 1$ alternatively $\int f_1(\vec{x}_1,\vec{p}_1)\,\mathrm{d}\vec{p}_1 = \frac{n(\vec{x}_1)}{n_0}$

$$f_2(\vec{x}_1,\vec{p}_1,\vec{x}_2,\vec{p}_2,t)= f_1(\vec{x}_1,\vec{p}_1,t)\, f_1(\vec{x}_2,\vec{p}_2,t)+g(\vec{x}_1,\vec{p}_1,\vec{x}_2,\vec{p}_2,t)$$

g - binary correlation function, 1st step in Mayer cluster expansion

internal force acting on particle is $\vec{F}=n_0\int \mathrm{d}\vec{x}_2\,\mathrm{d}\,\vec{p}_2\,\vec{F}_{12}\,f_1(\vec{x}_2,\vec{p}_2,t)$

$$\frac{\partial}{\partial t}f_1+\vec{\mathrm{v}}_1\nabla_{\vec{x}_1}f_1+\vec{F}\nabla_{\vec{p}_1}f_1=-n_0\int \mathrm{d}\vec{x}_2\,\mathrm{d}\,\vec{p}_2\vec{F}_{12}\nabla_{\vec{p}_1}g(\vec{x}_1,\vec{p}_1,\vec{x}_2,\vec{p}_2,t)$$

Term on the right side – collision integral $g = 0$ – Vlasov equation

Equation for g needed equation for f_2

$$\frac{\partial}{\partial t}f_2+\left(\vec{\mathrm{v}}_1\nabla_{\vec{x}_{1i}}+\vec{\mathrm{v}}_2\nabla_{\vec{x}_2}\right)f_2+\left(\vec{F}_{12}\nabla_{\vec{p}_1}+\vec{F}_{21}\nabla_{\vec{p}_2}\right)f_2=-n_0\int \mathrm{d}\vec{x}_3\,\mathrm{d}\,\vec{p}_3\left(\vec{F}_{13}\nabla_{\vec{p}_1}+\vec{F}_{23}\nabla_{\vec{p}_2}\right)f_3$$

Cluster expansion

$$f_3(1,2,3)= f_1(1)f_1(2)f_1(3)+ f_1(1)g(2,3)+ f_1(2)g(1,3)+ f_1(3)g(1,2)+h(1,2,3)$$

h – three-particle correlation function $\sim \Lambda^{-2}$ - will be omitted

$$\frac{\partial}{\partial t}g(1,2)+\left(\vec{\mathrm{v}}_1\nabla_{\vec{x}_{1i}}+\vec{\mathrm{v}}_2\nabla_{\vec{x}_2}\right)g(1,2)=-\left(\vec{F}_{12}\nabla_{\vec{p}_1}+\vec{F}_{21}\nabla_{\vec{p}_2}\right)\left[f_1(1)f_1(2)+g(1,2)\right]-$$

$$-\left\{n_0\int \mathrm{d}\vec{x}_3\,\mathrm{d}\,\vec{p}_3\vec{F}_{13}\nabla_{\vec{p}_1}\left[f_1(1)g(2,3)\right]+n_0\int \mathrm{d}\vec{x}_3\,\mathrm{d}\,\vec{p}_3\vec{F}_{23}\nabla_{\vec{p}_2}\left[f_1(2)g(1,3)\right]\right\}$$

Closed system of equations for f_1 and g_{12}

We want to express binary correlation function via one-particle dist. fun.

Homogeneous plasma without external fields and potential binary interaction assumed

$$f_1(\vec{x},\vec{p},t) = f_1(\vec{p},t) \qquad \vec{F}(\vec{x},t) = \vec{F}(t) = \vec{0} \quad \Rightarrow \quad g = g(\vec{x}_1 - \vec{x}_2, \vec{p}_1, \vec{p}_2, t)$$

$$\vec{F}_{12} = -\frac{\partial U_{12}(|\vec{x}_1 - \vec{x}_2|)}{\partial \vec{x}_1}$$

g is estimated from 1st term of equation $\dfrac{\Delta x}{\Delta t} \sim \mathrm{v}_T$

$$g_{12} \sim f_1 f_2 U_{12} / \varepsilon_T \qquad h_{123} \sim f_3 g_{12} U_{23} / \varepsilon_T$$ in integral meaning

Equation for f_1 and g_{12}, more sorts of particles

$$\frac{\partial f_a}{\partial t} + \vec{\mathrm{v}}_a \frac{\partial f_a}{\partial \vec{r}_a} + \tilde{\vec{F}}_a \frac{\partial f_a}{\partial \vec{p}_a} = \frac{\partial}{\partial \vec{p}_a} \sum_b \int \mathrm{d}\vec{r}_b \, \mathrm{d}\,\vec{p}_b \, g_{ab}(\vec{r}_a, \vec{p}_a, \vec{r}_b, \vec{p}_b, t) \frac{\partial U_{ab}(|\vec{r}_a - \vec{r}_b|)}{\partial \vec{r}_a}$$

Force consists of internal and external forces

$$\tilde{\vec{F}}_a = \vec{F}_a - \frac{\partial}{\partial \vec{r}_a} \sum_b \int \mathrm{d}\vec{r}_b \, \mathrm{d}\,\vec{p}_b \, f_b(\vec{r}_b, \vec{p}_b, t) \, U_{ab}(|\vec{r}_a - \vec{r}_b|)$$

We want to express **collision term**

$$\left[\frac{\partial f_a}{\partial t}\right]_c = \frac{\partial}{\partial \vec{p}_a}\sum_b \int \mathrm{d}\vec{r}_b \mathrm{d}\,\vec{p}_b\, g_{ab}(\vec{r}_a, \vec{p}_a, \vec{r}_b, \vec{p}_b, t)\,\frac{\partial U_{ab}(|\vec{r}_a - \vec{r}_b|)}{\partial \vec{r}_a}$$

Equation for g_{ab}

$$\left\{\frac{\partial}{\partial t} + \mathrm{v}_a\,\frac{\partial}{\partial \vec{r}_a} + \mathrm{v}_b\,\frac{\partial}{\partial \vec{r}_b} + \tilde{\vec{F}}_a\,\frac{\partial}{\partial \vec{p}_a} + \tilde{\vec{F}}_b\,\frac{\partial}{\partial \vec{p}_b}\right\} g_{ab} - \frac{\partial U_{ab}}{\partial \vec{r}_a}\left(\frac{\partial}{\partial \vec{p}_a} - \frac{\partial}{\partial \vec{p}_b}\right) g_{ab} =$$

$$= \left(\frac{\partial f_a}{\partial \vec{p}_a}\, f_b - f_a\,\frac{\partial f_b}{\partial \vec{p}_b}\right)\frac{\partial U_{ab}}{\partial \vec{r}_a} + \frac{\partial f_a}{\partial \vec{p}_a}\sum_c \int \mathrm{d}\vec{r}_c\, \mathrm{d}\,\vec{p}_c\, g_{bc}\,\frac{\partial U_{ac}}{\partial \vec{r}_a} + \{a \leftrightarrow b\} +$$

$$+ \sum_c \int \mathrm{d}\vec{r}_c\, \mathrm{d}\,\vec{p}_c\,\frac{\partial U_{ac}}{\partial \vec{r}_a}\,\frac{\partial}{\partial \vec{p}_a} h_{abc} + \{a \leftrightarrow b\}$$

<u>The task has 2 possible small parameters</u>

(1) **weak force interaction between particles**

$$\alpha = \frac{U_{ab}}{\varepsilon} \ll 1$$

ε - average kinetic energy

$$\Rightarrow \quad g_{ab} \sim f_a f_b \frac{U_{ab}}{\varepsilon} \ll f_a f_b$$

last term on the left side omitted compared to the 1st on right side

$$h_{abc} \sim f_a g_{cb} \frac{U_{bc}}{\varepsilon} \ll f_a g_{cb}$$

last term on the right << previous

(2) **short reach of forces** d – reach of forces, n – particle concentration

$$\beta = (n d^3) \frac{U_{ab}}{\varepsilon} \ll 1$$

2nd term on the right side omitted <<1st,

3rd term on the right side omitted <<1st.

in gases $U_{ab} \sim \varepsilon$ and $n d^3 \ll 1$ thus $\alpha \sim 1$, $\beta << 1$

in plasmas

$$\beta = (n r_D{}^3) \frac{e^2}{4\pi\varepsilon_0 r_D} \frac{1}{k_B T} \sim 1$$

and $\alpha << 1$

If $\alpha \ll 1 \wedge \beta \ll 1 \Rightarrow$ **Fokker-Planck** (Landau) collision term

simple for solving, assumptions are not fully met in any system

If $\beta \ll 1$ (any α) – **Boltzmann** collision term (gas, inelastic collisions in plasmas)

If $\alpha \ll 1$ (any β) – (Bogoljubov-)**Lenard-Balescu** collision term (elastic collisions in plasmas including large distance interactions)

How to express g_{ab}? Relaxation time τ_g of correlation function g (collision duration) << relaxation time τ_f of one-particle distribution (1/collision frequency)

It is enough to express g in the limit $t \to \infty$

Bogoljubov's hypothesis $g(t \to -\infty) = 0$ (particles are not correlated before collision, but correlation exists after collision $\Rightarrow$ **irreversibility** - time between collisions of the same 2 particles is big (information on previous collision is lost)

We derive Landau (Fokker-Planck) collision term – the simplest one

$$\left(\frac{\partial}{\partial t} + \vec{\mathrm{v}}_a \frac{\partial}{\partial \vec{r}_a} + \vec{\mathrm{v}}_b \frac{\partial}{\partial \vec{r}_b} \right) g_{ab} = \left(\frac{\partial f_a}{\partial \vec{p}_a} f_b - f_a \frac{\partial f_b}{\partial \vec{p}_b} \right) \frac{\partial U_{ab}\left(\left| \vec{r}_a - \vec{r}_b \right| \right)}{\partial \vec{r}_a}$$

What is $\boldsymbol{g}$ in **equilibrium**? Function f Maxwellian and time derivative= 0

$$f_a = \frac{n_a}{\left(2\pi m_a k_B T\right)^{3/2}} \exp\left(-\frac{p_a^2}{2m_a k_B T}\right) \Rightarrow g_{ab} = -\frac{1}{k_B T} U_{ab}\left(\left|\vec{r}_a - \vec{r}_b\right|\right) f_a f_b$$

(correct g for $U_{ab} << k_B T$)

Solution of the equation out of equilibrium

Solution in time t is found by integration from t_0, equation characteristics correspond to steady straight-line motion of particles a, b

$$\vec{r}_{a0}{}' = \vec{r}_a - \vec{\mathrm{v}}_a\left(t - t_0\right) \qquad \vec{r}_{at'}{}' = \vec{r}_a - \vec{\mathrm{v}}_a\left(t - t'\right)$$ and similarly, for particle b

$$g_{ab}(\vec{r}_a, \vec{p}_a, \vec{r}_b, \vec{p}_b, t) = g_{ab}(\vec{r}_{a0}{}', \vec{p}_a, \vec{r}_{b0}{}', \vec{p}_b, t_0) + \int_{t_0}^{t} dt' \left\{\frac{\partial}{\partial \vec{r}_a} U_{ab}\left(\left|\vec{r}_{at'}{}' - \vec{r}_{bt'}{}'\right|\right)\right\} \times$$

$$\times \left\{ f_b(\vec{r}_{bt'}{}', \vec{p}_b, t') \frac{\partial}{\partial \vec{p}_a} f_a(\vec{r}_{at'}{}', \vec{p}_a, t') - f_a(\vec{r}_{at'}{}', \vec{p}_a, t') \frac{\partial}{\partial \vec{p}_b} f_b(\vec{r}_{bt'}{}', \vec{p}_b, t') \right\}$$

spatially homogeneous f_a, f_b (at forces reach) constant in time (for collision duration)

$$g_{ab}(\vec{r}_a,\vec{p}_a,\vec{r}_b,\vec{p}_b,t)=g_{ab}(\vec{r}_{a0}{}',\vec{p}_a,\vec{r}_{b0}{}',\vec{p}_b,t_0)+\Big\{f_b(\vec{r}_b,\vec{p}_b,t)\frac{\partial}{\partial\vec{p}_a}f_a(\vec{r}_a,\vec{p}_a,t)-$$

$$-f_a(\vec{r}_a,\vec{p}_a,t)\frac{\partial}{\partial\vec{p}_b}f_b(\vec{r}_b,\vec{p}_b,t)\Big\}\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec{r}_a}U_{ab}\left(\left|\vec{r}_{at'}{}'-\vec{r}_{bt'}{}'\right|\right)$$

we use Bogoljubov’s hypothesis $t_0\to-\infty \quad g_{ab}(t\to-\infty)=0$

and into collision integral

$$\left(\frac{\partial f_a}{\partial t}\right)_c=\frac{\partial}{\partial\vec{p}_a}\sum_b\int \mathrm{d}\vec{r}_b\,\mathrm{d}\,\vec{p}_b\,g_{ab}(\vec{r}_a,\vec{p}_a,\vec{r}_b,\vec{p}_b,t)\frac{\partial U_{ab}(|\vec{r}_a-\vec{r}_b|)}{\partial\vec{r}_a}$$

we substitute expression for g_{ab} and following expression is obtained

$$\left(\frac{\partial f_a}{\partial t}\right)_c=\frac{\partial}{\partial p_{ai}}\sum_b\int \mathrm{d}\vec{p}_b\,I_{ij}^{ab}\left(\vec{\mathrm{v}}_a-\vec{\mathrm{v}}_b\right)\left(\frac{\partial f_a}{\partial p_{aj}}f_b-f_a\frac{\partial f_b}{\partial p_{bj}}\right)$$

where $$I_{ij}^{ab}(\vec{\mathrm{v}})=\int \mathrm{d}^3\,r_b\,\frac{\partial U_{ab}(|\vec{r}_a-\vec{r}_b|)}{\partial r_{ai}}\int_{-\infty}^{0}\mathrm{d}\,t\,\frac{\partial}{\partial r_{aj}}U_{ab}(|\vec{r}_a-\vec{r}_b+\vec{\mathrm{v}}\,t|)$$

We express the interaction potential using Fourier transformation

$U_{ab}(\vec{r}) = \int \frac{d^3 k}{(2\pi)^3} \Phi_{ab}(\vec{k}) \exp(i\vec{k}\vec{r})$, where $\Phi_{ab}(\vec{k}) = \frac{q_a q_b}{\varepsilon_0 k^2}$ is transform of Coulomb potential (derivation in appendix)

The integral diverges logarithmically → $k_{min}=1/r_{max}=1/r_D$, $k_{max}=1/r_{min}$

$$I_{ij}^{ab}(\vec{v}) = \frac{q_a^2 q_b^2}{8\pi\varepsilon_0} \frac{v^2 \delta_{ij} - v_i v_j}{v^3} \ln \Lambda_{ab},$$

where **Coulomb logarithm** $\ln \Lambda_{ab} = \ln(k_{max}/k_{min}) = \ln(r_D / r_{min})$

a $r_{min} = \max(b_0, \frac{\hbar}{2 m_{ab} v})$, where Landau length $b_0 = \frac{q_a q_b}{2\pi\varepsilon_0} \frac{1}{m_a v^2}$

$$\left(\frac{\partial f_a}{\partial t}\right)_c = \frac{\partial}{\partial p_{ai}} \sum_b \frac{q_a^2 q_b^2}{8\pi\varepsilon_0^2} \ln \Lambda_{ab} \int d\vec{p}_b \frac{v_{ab}^2 \delta_{ij} - v_{abi} v_{abj}}{v_{ab}^3} \left(\frac{\partial f_a}{\partial p_{aj}} f_b - f_a \frac{\partial f_b}{\partial p_{bj}} \right)$$

This can be written in form

$$\left(\frac{\partial f_a}{\partial t}\right)_c = -\frac{\partial}{\partial \vec{p}_a}\left[\vec{A}^{(a)} f_a\right] + \frac{\partial}{\partial p_{ai}}\left[D_{ij}^{(a)} \frac{\partial f_a}{\partial p_{aj}}\right]$$

Dynamic friction force (it decelerates moving particles)

$$A_i^{(a)}(\vec{p}_a) = \sum_b \frac{q_a^2 q_b^2}{8\pi\varepsilon_0^2} \ln\Lambda_{ab} \int \mathrm{d}\vec{p}_b \, \frac{\mathrm{v}_{ab}^2\,\delta_{ij} - \mathrm{v}_{abi}\,\mathrm{v}_{abj}}{\mathrm{v}_{ab}^3} \frac{\partial f_b}{\partial p_{bj}}$$

Coefficient of diffusion in momentum space (spreads particle beam)

$$D_{ij}^{(a)}(\vec{p}_a) = \sum_b \frac{q_a^2 q_b^2}{8\pi\varepsilon_0^2} \ln\Lambda_{ab} \int \mathrm{d}\vec{p}_b \, \frac{\mathrm{v}_{ab}^2\,\delta_{ij} - \mathrm{v}_{abi}\,\mathrm{v}_{abj}}{\mathrm{v}_{ab}^3} f_b$$

Other form is also used

$$\left(\frac{\partial f_a}{\partial t}\right)_c = -\frac{\partial}{\partial \vec{p}_a}\left[\tilde{\vec{A}}_{(a)} f_a\right] + \frac{1}{2}\frac{\partial}{\partial p_{ai}}\frac{\partial}{\partial p_{aj}}\left[B_{ij}^{(a)} f_a\right]$$

where coefficients are expressed via sums $\tilde{\vec{A}}_{(a)} = \sum_b \tilde{\vec{A}}_{ab} \qquad B_{ij} = \sum_b B_{ij}^{ab}$

and partial terms may be expressed via Rosenbluth potentials ***G*** a ***H***

$$\tilde{\vec{A}}_{ab} = \Psi^{a,b} \frac{\partial}{\partial \vec{p}} H^{ab}(\vec{p}) \qquad\qquad B_{ij}^{ab} = \Psi^{a,b} \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j} G^{ab}(\vec{p})$$

Lenard-Balescu collision term

$\int$ of g_{ac} and g_{bc} are preserved – they describe long-distance correlations – collision term includes **dynamic shielding** of collisions by plasma

Nearly the same form as FP, but kernel

$$I_{ij}^{ab} = \pi \int \frac{\mathrm{d}\vec{k}}{(2\pi)^3} \left(\frac{q_a q_b}{\varepsilon_0 k^2} \right)^2 \frac{k_i k_j \delta\left[\vec{k}\left(\vec{\mathrm{v}}_a - \vec{\mathrm{v}}_b\right)\right]}{\left|\varepsilon^l\left(\vec{k}\,\vec{\mathrm{v}}_a, \vec{k}\right)\right|^2}$$

ε^l – longitudinal relative permittivity, when $\varepsilon^l = 1 \Rightarrow$ Fokker-Planck (Landau)

$$\varepsilon^l\left(\omega, \vec{k}\right) = 1 + \sum_a \frac{q_a^2}{\varepsilon_0 k^2} \int \mathrm{d}\vec{p}_a \, \frac{1}{\omega - \vec{k}\,\vec{\mathrm{v}}_a + i\Delta} \, \vec{k} \, \frac{\partial f_a}{\partial \vec{p}_a}$$

Boltzmann collision term

Last term on the right side is preserved during of g derivation – characteristics do not correspond to straight-line motion, but to motion in field g in equilibrium (f is stationary homogeneous Maxwell's distribution)

$$g_{ab} = \left[\exp\left(-\frac{U_{ab}\left(\left|\vec{r}_a - \vec{r}_b\right|\right)}{k_B T} \right) - 1 \right] f_a f_b$$

for α<<1 result is the same as that for Fokker-Planck equation

Boltzmann collision term

$$\left(\frac{\partial f_a}{\partial t}\right)_c = \sum_b \int \mathrm{d}\vec{p}_b \, \mathrm{v}_{ab} \left[f_a(\vec{p}_a{}') f_b(\vec{p}_b{}') - f_a(\vec{p}_a) f_b(\vec{p}_b) \right] \mathrm{d}\sigma_{ab}(\mathrm{v}_{ab}, \theta, \varphi)$$

for elastic spheres of radius a je $\mathrm{d}\sigma = a^2 \, \mathrm{d}\Omega$

pro $\mathrm{U} \sim 1/\mathrm{r}^n \quad \Rightarrow \quad \mathrm{d}\sigma \sim \mathrm{d}\Omega / \mathrm{v}^{4/n}$

often used for real gas

Lennard-Jones potential $\mathrm{U} = 4\varepsilon\left[(a/r)^{12} - (a/r)^6 \right]$

for Coulomb potential

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left[\frac{q_a q_b}{8\pi\varepsilon_0 \, m_{ab} \, \mathrm{v}_{ab}^2} \right]^2 \frac{1}{\sin^4(\theta/2)}$$

Apendix – Derivation of kernel of Fokker-Planck collision integral

First, Fourier transformation of Coulomb potential is calculated

$$U_{ab}(r)=\frac{q_a q_b}{4\pi\varepsilon_0\, r} \quad\Rightarrow\quad \Phi_{ab}(\vec{k})=\int U_{ab}(r)\exp(-i\vec{k}\vec{r})\,\mathrm{d}\vec{r}=\frac{q_a q_b}{4\pi\varepsilon_0}\int\frac{\mathrm{e}^{-i\vec{k}\vec{r}}}{r}\,\mathrm{d}\vec{r}$$

$$\iiint\frac{\mathrm{e}^{-ikr\cos\theta}}{r}r^2\mathrm{d}r\,\mathrm{d}\varphi\sin\theta\,\mathrm{d}\theta=2\pi\int r\;\mathrm{d}r\int_{-1}^{1}\mathrm{e}^{ikry}\;\mathrm{d}y=\frac{4\pi}{k}\int_0^\infty\sin kr\,\mathrm{d}r=\frac{4\pi}{k^2}$$

and thus

$$\Phi_{ab}(\vec{k})=\frac{q_a q_b}{\varepsilon_0\, k^2}$$

$$U_{ab}(\vec{r})=\int\frac{\mathrm{d}\vec{k}}{(2\pi)^3}\Phi_{ab}(\vec{k})\exp(i\vec{k}\vec{r}) \qquad \frac{\partial U_{ab}(\vec{r})}{\partial r_j}=i\int k_j\,\frac{\mathrm{d}\vec{k}}{(2\pi)^3}\Phi_{ab}(\vec{k})\exp(i\vec{k}\vec{r})$$

$$\int_{-\infty}^{0}\mathrm{d}\,t\,\frac{\partial}{\partial r_{aj}}U_{ab}(|\vec{r}_a-\vec{r}_b+\vec{\mathrm{v}}\,t|)=\int_{-\infty}^{0}\mathrm{d}t\;\;i\int k_j\,\frac{\mathrm{d}\vec{k}}{(2\pi)^3}\Phi_{ab}(\vec{k})\,\mathrm{e}^{i\vec{k}(\vec{r}_a-\vec{r}_b)+i\vec{k}\,\vec{\mathrm{v}}\,t}=$$

$$=i\int k_j\,\frac{\mathrm{d}\vec{k}}{(2\pi)^3}\Phi_{ab}(\vec{k})\,\mathrm{e}^{i\vec{k}(\vec{r}_a-\vec{r}_b)}\overbrace{\int_{-\infty}^{0}\mathrm{d}t\;\,\mathrm{e}^{i\vec{k}\,\vec{\mathrm{v}}\,t}}^{\pi\,\delta(\vec{k}\,\vec{\mathrm{v}})}=i\pi\int k_j\,\frac{\mathrm{d}\vec{k}}{(2\pi)^3}\Phi_{ab}(\vec{k})\,\mathrm{e}^{i\vec{k}(\vec{r}_a-\vec{r}_b)}\,\delta\left(\vec{k}\,\vec{\mathrm{v}}\right)$$

Apendix

$$\int d\vec{r}_b \; i\int k_i' \frac{\mathrm{d}\vec{k}'}{(2\pi)^3}\Phi_{ab}(\vec{k}')\,\mathrm{e}^{i\vec{k}'(\vec{r}_a-\vec{r}_b)}\; i\pi\int k_j \frac{\mathrm{d}\vec{k}}{(2\pi)^3}\Phi_{ab}(\vec{k})\,\mathrm{e}^{i\vec{k}(\vec{r}_a-\vec{r}_b)}\,\delta\left(\vec{k}\,\vec{\mathrm{v}}\right)=$$

$$=-\int k_j \frac{\mathrm{d}\vec{k}}{(2\pi)^3}\Phi_{ab}(\vec{k})\,\delta\left(\vec{k}\,\vec{\mathrm{v}}\right)\underbrace{\int k_i' \frac{\mathrm{d}\vec{k}'}{(2\pi)^3}\Phi_{ab}(\vec{k}')\underbrace{\int d\vec{r}_b\;\mathrm{e}^{i(\vec{k}+\vec{k}')(\vec{r}_a-\vec{r}_b)}}_{=(2\pi)^3\delta(\vec{k}+\vec{k}')}}_{=-k_i\Phi(-\vec{k})}=$$

$$=\pi\int\frac{\mathrm{d}\vec{k}}{(2\pi)^3}k_ik_j\,\Phi^2_{ab}(k)\;\delta\left(\vec{k}\,\vec{\mathrm{v}}\right)\qquad\left(\Phi(-\vec{k})=\Phi^*(\vec{k})=\Phi(k)\right)$$

without any limitation, one can assume $\vec{\mathrm{v}}=(\mathrm{v},0,0)$, then relations hold

$$\int k_x\Phi^2(k)\,\delta(k_x\mathrm{v})\,\mathrm{d}k_x=0\quad\wedge\quad\int k_x^2\,\Phi^2(k)\,\delta(k_x\mathrm{v})\,\mathrm{d}k_x=0$$

and integral = 0 for $i=x\vee j=x$, mixed *yz* component is also = 0

$$I_{zz}=I_{yy}=\pi\iint\frac{\mathrm{d}k_y\mathrm{d}k_z}{(2\pi)^3}k_y^2\int\mathrm{d}k_x\Phi^2_{ab}(k)\,\delta\left(k_x\mathrm{v}\right)=\pi\iint\frac{\mathrm{d}k_y\mathrm{d}k_z}{(2\pi)^3}\frac{k_y^2}{\mathrm{v}}\Phi^2_{ab}(\sqrt{k_y^2+k_z^2})=$$

$$=\frac{\pi}{\mathrm{v}}\int_0^\infty\int_0^{2\pi}\frac{k\,\mathrm{d}k\,\mathrm{d}\varphi}{(2\pi)^3}\left(\frac{q_aq_b}{\varepsilon_0k^2}\right)^2k^2\sin^2\varphi=\frac{q_a^2q_b^2}{8\pi\,\varepsilon_0^2\,\mathrm{v}}\int_0^\infty\frac{\mathrm{d}k}{k}\triangleq\frac{q_a^2q_b^2}{8\pi\,\varepsilon_0^2\,\mathrm{v}}\ln\Lambda_{ab}$$

and now we only have to transform for any direction of velocity as the only direction is $\vec{\mathrm{v}}$, there can be used only tensors δ_{ij} and $\mathrm{v}_i\mathrm{v}_j/\mathrm{v}^2$

for $\vec{\mathrm{v}} = (\mathrm{v}, 0, 0)$ is

$$I_{ij} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij} - \frac{\mathrm{v}_i \, \mathrm{v}_j}{\mathrm{v}^2}$$

$$\Rightarrow \quad I_{ij}^{ab}(\mathrm{v}) = \frac{q_a^2 q_b^2}{8\pi\,\varepsilon_0^2} \frac{\mathrm{v}^2\,\delta_{ij} - \mathrm{v}_i\,\mathrm{v}_j}{\mathrm{v}^3} \ln \Lambda_{ab}$$