Diffusion

Fick's law of diffusion in gas

Let's assume motionless ideal gas of constant temperature T and constant pressure p, in which there is a low inhomogeneous mass concentration of admixture α. Then the total force on admixture α in stationary state must be zero

$$0 = -m_\alpha n_\alpha \nu_{\alpha p} \vec{u}_\alpha - \nabla p_\alpha = -m_\alpha n_\alpha \nu_{\alpha p} \vec{u}_\alpha - k_B T \nabla n_\alpha \quad ,$$

where $\nu_{\alpha p}$ is collision frequency of admixture α with gas molecules.

The flux of particles α is $\vec{\Gamma}_\alpha = n_\alpha \vec{u}_\alpha = -D_\alpha \nabla n_\alpha = -\dfrac{k_B T}{m_\alpha \nu_{\alpha p}} \nabla n_\alpha$ ***Fick's law***

After substitution into continuity equation

$$\frac{\partial n_\alpha}{\partial t} = \mathrm{div}\left(D_\alpha \nabla n_\alpha\right) = D_\alpha \Delta n_\alpha \quad ,$$

as the diffusion coefficient is constant here

Parabolic partial differential equation $L^2 = D\tau$

Diffusion in weakly ionized gas

$n_n =>$ $\lambda_f = \dfrac{1}{n_n \sigma}$ $\nu = n_n \langle \sigma \mathrm{v} \rangle = \left\langle \dfrac{\mathrm{v}}{\lambda_f} \right\rangle$

Equations of motion (without or along magnetic field)

$$m_\alpha n_\alpha \left[\frac{\partial \vec{u}_\alpha}{\partial t} + (\vec{u}_\alpha \nabla) \vec{u}_\alpha \right] = q_\alpha n_\alpha \vec{E} - m_\alpha n_\alpha \nu_{\alpha n} \vec{u}_\alpha - \nabla p_\alpha$$

α is the particle type, stationary state. $\dfrac{\partial \vec{u}_\alpha}{\partial t} = 0$, term $(\vec{u}_\alpha \nabla) \vec{u}_\alpha$ omitted (quadratic)

$$\vec{u}_\alpha = \frac{q_\alpha}{m_\alpha \nu_{\alpha n}} \vec{E} - \frac{k_B T_\alpha}{m_\alpha \nu_{\alpha n}} \cdot \frac{\nabla n_\alpha}{n_\alpha} \qquad \vec{\Gamma}_\alpha = n_\alpha \vec{u}_\alpha = \pm \mu_\alpha n_\alpha \vec{E} - D_\alpha \nabla n_\alpha$$

μ_e, μ_i are mobilities $\mu_\alpha = \left| \dfrac{q_\alpha}{m_\alpha \nu_{\alpha n}} \right|$ D_e, D_i diffusion coefficients $D_\alpha = \dfrac{k_B T_\alpha}{m_\alpha \nu_{\alpha n}}$

Ambipolar diffusion – during diffusion electric field arises to secure quasineutrality

charge must stay ≈ 0 => $$q_e\vec{\Gamma}_e + q_i\vec{\Gamma}_i = 0 \qquad q_e n_e + q_i n_i \simeq 0$$

Weakly ionized gas $Z = 1$ $q_i = e$ $q_e = -e$ $\Gamma_e = \Gamma_i = \Gamma$ $n_e = n_i = n$

$$\vec{\Gamma} = \mu_i n\vec{E} - D_i\nabla n = -\mu_e n\vec{E} - D_e\nabla n \quad \Rightarrow \quad \vec{E} = \frac{D_i - D_e}{\mu_i + \mu_e}\frac{\nabla n}{n}$$

Electric field is proportional to the density gradient and ambipolar diffusion coefficient D_a

$$\vec{\Gamma} = -\frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}\nabla n = -D_a\nabla n$$

In plasma without B is $\mu_e >> \mu_i$ and

$$D_a \approx D_i + \frac{T_e}{T_i}D_i \qquad (\approx 2D_i \text{ at equal temperatures of } e \text{ and } i)$$

Diffusion in direction perpendicular to B

In **weakly ionized gas** ν is collision frequency of studied particles with neutrals, let B is in direction of axis z and density gradient is in direction of x axis, and let $\tau = \nu^{-1}$

$$0 = -k_B T \frac{\partial n}{\partial x} + nqu_y B - \nu nmu_x \qquad\qquad 0 = -nqu_x B - \nu nmu_y$$

Then

$$u_x = -\frac{k_B T}{nm\nu\left(1 + \frac{\omega_c^2}{\nu^2}\right)} \frac{\partial n}{\partial x} \quad \Rightarrow \quad \Gamma_\perp = -\frac{k_B T}{m\nu\left(1+\omega_c^2\tau^2\right)} \frac{\partial n}{\partial x} = -\frac{D_{||}}{\left(1+\omega_c^2\tau^2\right)} \frac{\partial n}{\partial x}$$

$$D_\perp = \frac{D_{||}}{1+\omega_c{}^2\tau^2}$$

If $\omega_c \tau = \frac{\omega_c}{\nu} = \frac{\lambda_f}{r_L} \gg 1$, then $D_\perp = \frac{D}{\omega_c{}^2\tau^2} = \frac{k_B T\nu}{m\omega_c{}^2} = \frac{mk_B T\nu}{q^2 B^2}$

Diffusion coefficient across B is inversely proportional to B^2 and directly proportional to the collision frequency ν; without collisions there would be no diffusion. After a collision, the particle moves by 2 Larmor radii at maximum; thus, the Larmor radius (gyroradius) is substituted for the mean free path.

Across B ions are more mobile than electrons – however, electric field generated during ambipolar diffusion cannot influence electron and ion fluxes in the direction of electric field.

Additionally, particle flux arises in the direction normal to B and ∇n, it is diamagnetic drift that we derived earlier (chapter 4) disregarding collisions $(\omega_c \tau \gg 1)$.

If plasma column is not very long, quasineutrality can be reached by longitudinal electron current to anode

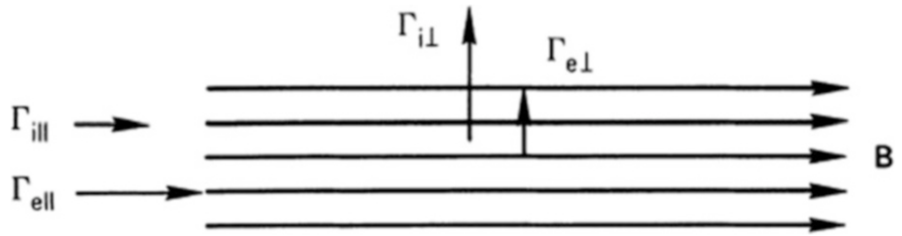
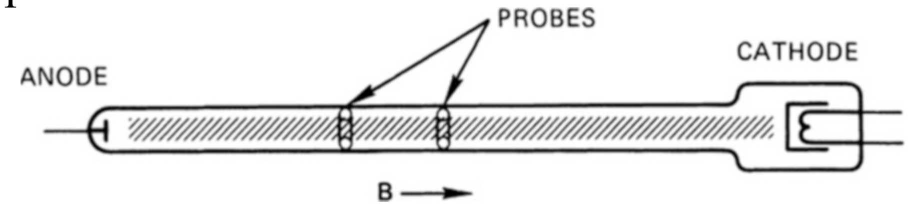
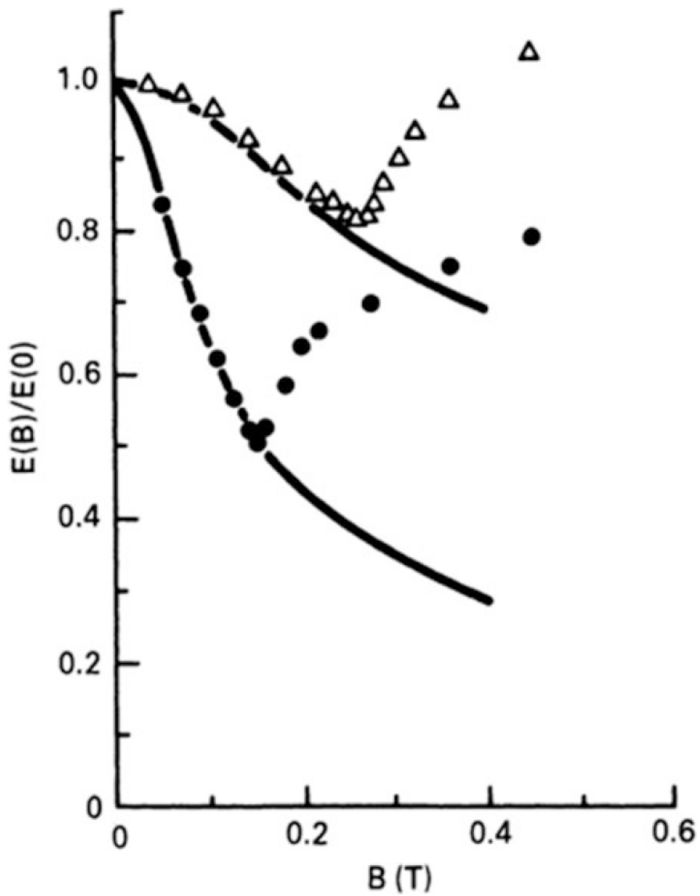


Fig. Parallel and perpendicular particle fluxes in a magnetic field

Very long plasma column – stationary self-sustained positive column of the discharge - charged particle loss due to the transverse diffusion is balanced by collisional ionization



The **Lehnert-Hoh experiment** (1960) to check the effect of a magnetic field on diffusion in a weakly ionized gas (3.5 m long positive column of diameter 1 cm)



Assumption: transverse diffusion will decrease with increasing longitudinal magnetic field, and thus electric field can be smaller to cause sufficient ionization rate

Experimental results for 2 values of pressure

Experimental result agrees with the assumption for low values of ***B***. However, the situation qualitatively changes when threshold value of ***B*** is surpassed.

It was explained (Kadomtsev and Nedospasov) by an instability leading to helical distortion of the positive column and increased transverse diffusion.

Fully ionized plasma,

The collision term for electrons will be $\nu_{ei} n_e m_e (\vec{u}_i - \vec{u}_e)$, diffusion velocities of electrons and ions will be equal, ambipolar field does not occur (and morever field in the direction of the density gradient does not influence diffusion velocity in this direction)

$$D_\perp = \frac{k_B (T_e + T_i / Z) \nu_{ei}}{m_e \omega_c^2} = \frac{m_e k_B (T_e + T_i / Z) \nu_{ei}}{e^2 B^2} \sim n T^{-1/2} B^{-2}$$

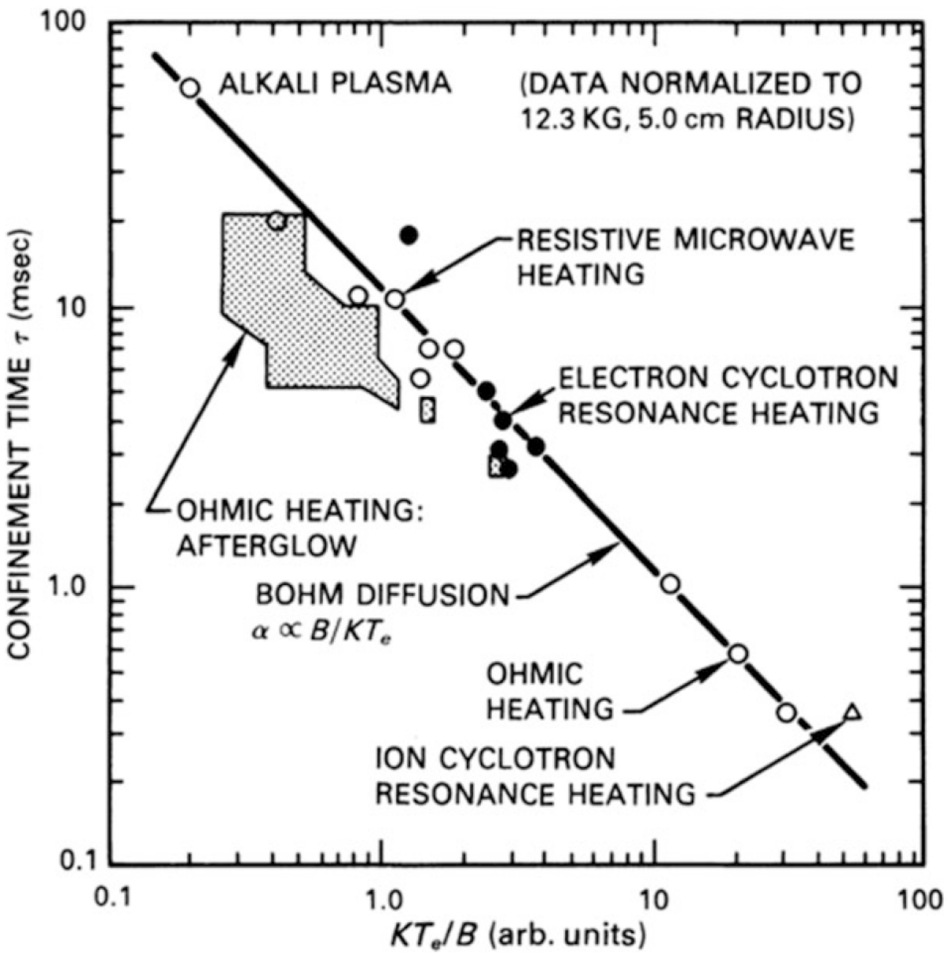
Classic collisional diffusion is proportional to B^{-2}, however, diffusion during magnetic confinement (in tokamaks etc.) is often higher, proportional to B^{-1}

Coefficient of **Bohm diffusion** was deduced from experimental results $D_\perp \simeq \frac{1}{16} \frac{kT_e}{eB}$

Various explanations- 1) Defects of magnetic field– possibility of field lines going to wall

2) Asymmetric electric field – asymmetry of vacuum chamber or asymmetry of plasma generation or heating ⇒ *E*x*B* drift - convective cells

3) Instabilities leading to the generation of plasma waves that produce oscillating electric fields and *E*x*B* drifts



Summary of confinement time measurements taken on various types of discharges in the Model C Stellarator, showing adherence to the Bohm diffusion law

Bohm formula is a natural scaling for losses due to *E*x*B* drift, where flux is expressed

$$\Gamma_\perp = n v_\perp \propto nE/B$$

and the electric field is estimated

$$E_{\max} \approx \frac{\phi_{\max}}{R} \approx \frac{KT_e}{eR}$$

and

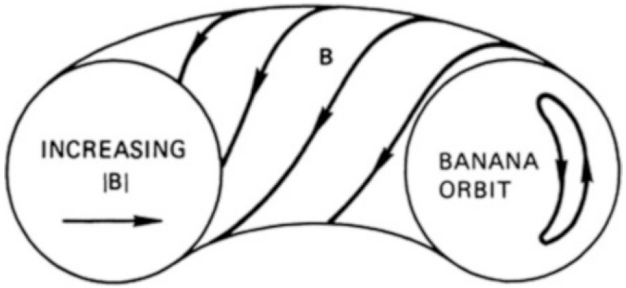
$$\Gamma_\perp \approx \gamma \frac{n}{R} \frac{KT_e}{eB} \approx -\gamma \frac{KT_e}{eB} \nabla n = -D_B \nabla n$$

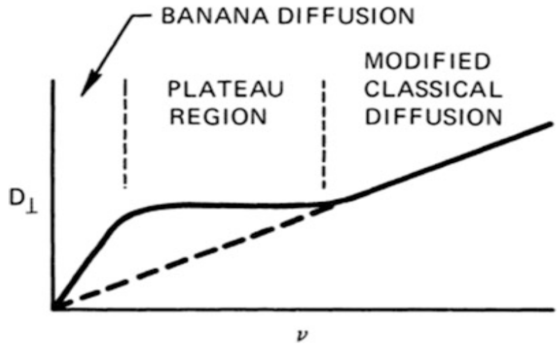
Bohm diffusion is typically by 4 orders higher than classical for magnetic confinement

Accurate configuration of fields may bring the diffusion closer to the classic collisional limit.

Chen writes - values 1000 times smaller have been already achieved in magnetic confinement

Diffusion can be increased in toroidal magnetic vessels due to existence of elongated closed orbits („banana orbit“) and this is called **neoclassical diffusion**.





Comparison of the values of Bohm and classical diffusion coefficients

Parameterss: $T = 100$ eV, $B = 1$ T, $Z = 1$, $n_e = n_i = 10^{19}$ m^{-3}, assumed $\ln\Lambda_{ei} = 10$

$$D_{\mathrm{B}} = \frac{1}{16}\frac{(10^2)(1.6\times10^{-19})}{(1.6\times10^{-19})(1)} = 6.25\mathrm{m}^2/\mathrm{s}$$

Electron-ion collision frequency

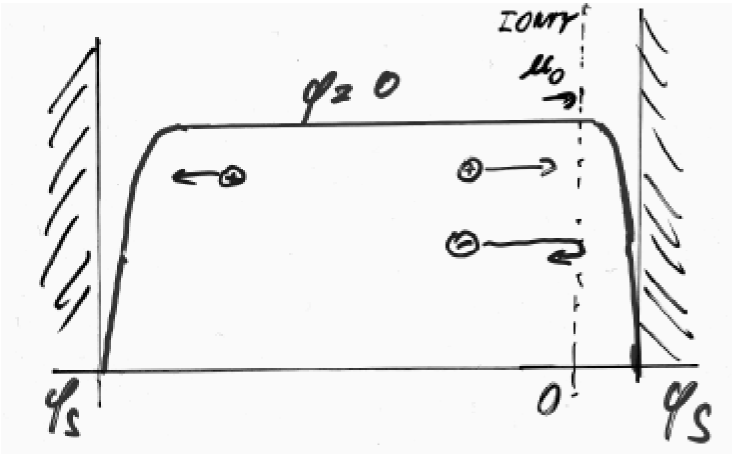
$$\nu_{ei} = \frac{4}{3}\frac{\sqrt{2\pi}\, q_e^2\, q_i^2\, n_i}{\sqrt{m_e}\,(4\pi\varepsilon_0)^2\,(k_B T_e)^{3/2}}\ln\Lambda_{ei} \qquad \nu_{ei} = 2.91\times10^5\ \mathrm{s}^{-}1 \qquad \omega_c = 1.76\times10^{11}\ \mathrm{s}^{-}1 \qquad \omega_c\tau = \omega_c/\nu_{ei} = 6.05\times10^5$$

Classical diffusion coefficient is ~10^4 times smaller than for Bohm diffusion

$$D_\perp = \frac{m_e\, k_B T_e\, \nu_{ei}}{q_e^2\, B^2} = 3.31\times10^{-4}\ \mathrm{m}^2/\mathrm{s}$$

Areas without quasineutrality

Sheath (near-wall layer)



Recombination on walls

Σ charge flux on walls = 0

Ions with mean ion charge Z = 1 –
assumed cold with velocity u_0 for $\varphi = 0$

$$u = \sqrt{{u_0}^2 - \frac{2e\varphi}{m_i}} \qquad \Big| \; \varphi < 0$$

$$n_i\left(x\right) = \frac{n_0 u_0}{u\left(x\right)} = \frac{n_0}{\sqrt{1 - \dfrac{2e\varphi}{m_i {u_0}^2}}}$$

Electrons – in equilibrium with field

$$n_e\left(x\right) = n_0 \exp\left(\frac{e\varphi}{k_B T_e}\right)$$

Poisson equation

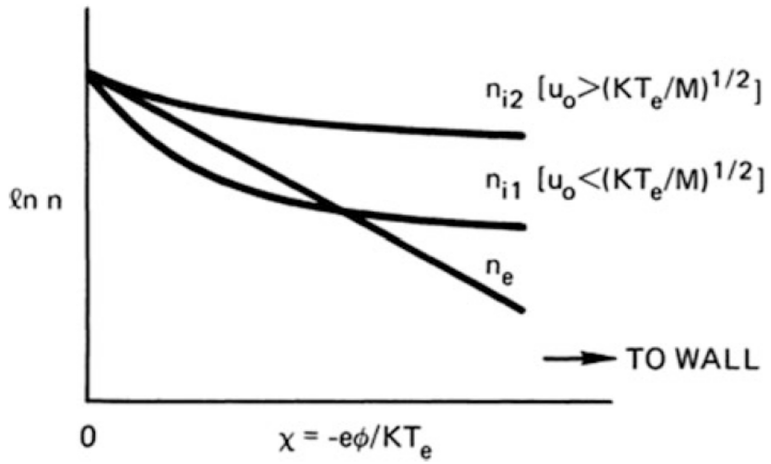
$$\frac{d^2\varphi}{dx^2} = \frac{e}{\varepsilon_0}\left(n_e - n_i\right) = \frac{en_0}{\varepsilon_0}\left[\exp\left(\frac{e\varphi}{k_B T_e}\right) - \frac{1}{\sqrt{1 - \frac{2e\varphi}{m_i u_0}}}\right]$$

is transformed to dimensionless coordinates

$$\chi = -\frac{e\varphi}{k_B T_e} \qquad \xi = \frac{x}{\lambda_D} \qquad M = \frac{u_0}{\sqrt{k_B T_e / m_i}} = \frac{u_0}{c_s}$$

$$\chi'' = \left(1 + \frac{2\chi}{M^2}\right)^{-1/2} - e^{-\chi}$$ we multiply equation ×χ´ and integrate $\int_0^\xi$

$$\frac{1}{2}\left(\chi'^2 - \chi_0'^2\right) = M^2\left[\left(1 + \frac{2\chi}{M^2}\right)^{1/2} - 1\right] + e^{-\chi} - 1$$ in point 0 is $E \approx 0$ and thus $\chi_0' \approx 0$



Sheath keeps away electrons and attracts ions

$\Rightarrow \quad n_i > n_e$

Plane neighborhood, where plasma enters sheath, is derived from Taylor expansion for $\chi \approx 0$
terms proportional to 1 and χ cancel, the first ~ χ^2

$$\frac{1}{2}\chi'^2 \approx \frac{1}{2}\chi^2\left(-\frac{1}{M^2}+1\right) > 0$$

possible only for M > 1

Bohm criterion – stationary solution exists only for $u_0 > c_s$

How to find the potential φ_s of the wall?

$$\frac{1}{2}\mathrm{v}_{T_e} \cdot n_0 \exp(\frac{e\varphi_s}{k_B T_e}) \approx n_0 u_0 \qquad \varphi_s \approx \frac{k_B T_e}{e}\ln\frac{2u_0}{\mathrm{v}_{T_e}}$$

In the wall neighborhood $n_e << n_i,\ 2\chi >> M^2 \quad \Rightarrow \quad \chi'' = \left(1+\frac{2\chi}{M^2}\right)^{-1/2} \approx \frac{M}{\sqrt{2\chi}}$

after integration (z denotes place where one can assume $n_e \approx 0$)

$$\frac{1}{2}\left(\chi'^2 - \chi_z'^2\right) = \sqrt{2}M\left(\chi^{1/2} - \chi_z^{1/2}\right)$$

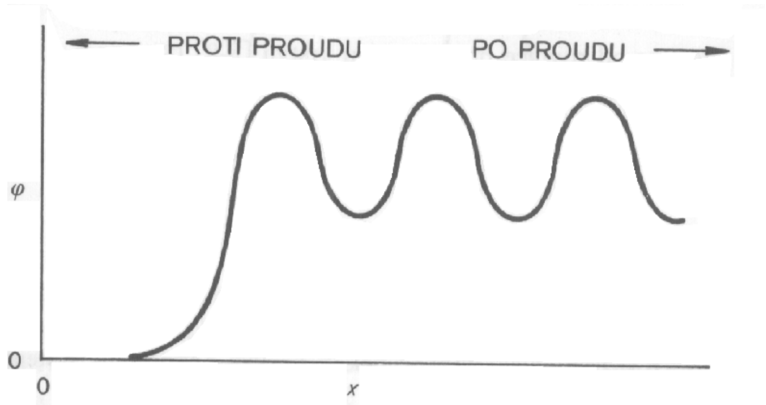
on the assumption $\chi_z \approx 0$, $\chi_z' \approx 0$ $\qquad \chi' = 2^{3/4} M^{1/2} \chi^{1/4}$

wall position is $\xi_s = \xi_z + \xi_d$ (ξ_d is the layer thickness $n_e \approx 0$)

then the relation holds $M = \dfrac{4\sqrt{2}}{9}\dfrac{\chi_s^{3/2}}{\xi_d^{\,2}}$ $\Rightarrow$ $J = en_0u_0 = \dfrac{4}{9}\left(\dfrac{2e}{m_i}\right)^{1/2}\dfrac{\varepsilon_0 \left|\varphi_s\right|^{3/2}}{d^2}$

Child-Langmuir law

Similar sheath is



collisionless ion-acoustic shock wave
(collisionless shock)

Sagdeev's potential