Ion-sound waves (electrostatic low frequency waves)

are longitudinal waves similar classical sound

in gas
$$c_s = \frac{\omega}{k} = \left(\frac{\gamma k_B T}{M}\right)^{1/2}$$

plasma – sound is slow for electrons, but fast for ions

Electron density is in each moment in equilibrium with instantaneous potential:

$$n_e = n_0 \exp\left(\frac{e\Phi_1}{k_B T_e}\right) = n_0 \left(1 + \frac{e\Phi_1}{k_B T_e} + \ldots\right)$$

$$n_1 = n_0 \frac{e\Phi_1}{k_B T_e} \qquad \text{electrons are isothermal } (\gamma = 1)$$

Electric field may be also derived from quasi-neutrality condition–sum of forces acting on electrons $= 0$

$$eE_1 = -e\nabla\Phi_1 = -\frac{1}{n_0}\nabla p_{e1} = -\frac{k_B T_e}{n_0}\nabla n_1 = -ikk_B T_e \frac{n_1}{n_0} \qquad \Rightarrow E_1$$

hydrodynamic equation for ion fluid

$$i\omega n_{i1} = n_{i0} ik \mathrm{v}_{i1}$$

$$n_{i0} = \frac{n_0}{Z}$$

$$-i\omega M \mathrm{v}_{i1} = -\frac{\nabla p_{i1}}{n_{i0}} + ZeE_1$$

$$E_1 = -\nabla\Phi = -ik\Phi_1$$

Poisson equation is not needed

$$\cancel{-\varepsilon_0 \Delta\Phi_1 = Zen_{i0} - en_1 = e\left(Zn_{i1} - n_1\right)}$$

slow motion $\rightarrow$ quasineutrality $\quad Zn_{i1} = n_1$

we express $\quad \Phi_1 = \dfrac{k_B T_e}{en_0} n_1 = \dfrac{k_B T_e}{en_0} Zn_{i1} \quad \Rightarrow$ equations of motion

$$-i\omega M n_{i0} \mathrm{v}_{i1} = -ik\gamma_i k_B T_i n_{i1} - ikk_B T_e Zn_{i1} \quad \text{- ion motion}$$

$$i\omega n_{i1} = n_{i0} ik \mathrm{v}_{i1} \qquad \text{continuity equation}$$

$\Rightarrow$ dispersion equation

$$\omega^2 = \left(\underbrace{\frac{\gamma_i k_B T_i + Z k_B T_e}{M}}_{c_s{}^2} \right) \cdot k^2$$

usually ions are adiabatic $\gamma_i = 5/3$

If $ZT_e \approx T_i$, then there is strong collisionless damping by ions,
phase velocity $c_s \approx$ ion thermal velocity

Ion-sound waves for $ZT_e \gg T_i$ are weakly damped

ion-sound velocity $\qquad c_s \simeq \sqrt{\dfrac{Z k_B T_e}{M}}$

The above applied plasma approximation does not hold for large k comparable with λ_{De}^{-1}. Consequently, dispersion relation will be derived without plasma approximation:

$$\nabla \vec{E}_1 = -\Delta \Phi_1 = k^2 \Phi_1 = e\left(Zn_{i1} - n_{e1}\right)/\varepsilon_0$$

$$n_{e1} = \frac{e\Phi_1}{k_B T_e} n_0$$

Density perturbation is included into Poisson equation

$$\Phi_1 \left(k^2 + \frac{n_0 e^2}{\varepsilon_0 k_B T_e} \right) = \frac{Zen_{i1}}{\varepsilon_0}$$

$$\Phi_1 = \frac{Zen_{i1}}{\varepsilon_0} \frac{\lambda_{De}^2}{1 + k^2 \lambda_{De}^2}$$

potential Φ_1 is included into ion equation of motion

$$\frac{\omega}{k} = \left(\frac{Zk_B T_e}{M} \frac{1}{1 + k^2 \lambda_{De}^2} + \frac{\gamma_i k_B T_i}{M} \right)^{\!1/2}$$

Dispersion relation of ion-sound waves differs due to inclusion of deviation from quasineutrality only by the term $k^2 \lambda_{De}^{\;2}$,

The most simple relation $k >> \lambda_{De}^{\;-1}$ a $T_i = 0$

$$\omega^2 = \frac{n_0 Z e^2}{\varepsilon_0 M} = \frac{n_{i0} Z^2 e^2}{\varepsilon_0 M} = \omega_{pi}^{\;2}$$

… ion plasma frequency

Elmg. waves in plasmas without external magnetic field B_0

Maxwell's equations

we transform to wave equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{j}$$

$$\nabla \times \left(\nabla \times \vec{E} \right) + \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial \vec{j}}{\partial t}$$

Plasma impact via high-frequency current density $\vec{j}$ (electron current dominant)
Fluid equations for electrons – equation for electron fluid velocity

$$\frac{\partial \vec{v}_e}{\partial t} + \left(\vec{v}_e \nabla \right) \vec{v}_e = -\frac{1}{n_e m_e} \nabla p_e - \frac{e}{m_e} \left(\vec{E} + \vec{v}_e \times \vec{B} \right)$$

A linear (i.e. weak) electromagnetic wave assumed, linearization

$$\vec{E} = \vec{E}_1 \quad \vec{B} = \vec{B}_1 \quad \vec{j} = \vec{j}_1 \quad \vec{v}_e = \vec{v}_1 \quad n_e = n_0 + n_1 \quad p_e = p_0 + p_1$$

Magnetic force $-e \left(\vec{v}_1 \times \vec{B}_1 \right)$ omitted (small of the 2nd order)

$$\vec{v}_1 \| \vec{E}_1, \quad \mathrm{div}\vec{E}_1 = 0 \Rightarrow \mathrm{div}\vec{v}_1 = 0, \, n_1 = 0 \Rightarrow p_1 = 0$$

electron density does not change, equation for electron velocity simplified

$$\frac{\partial \vec{v}_{e1}}{\partial t} = -\frac{e}{m_e}\vec{E}_1$$

current density

$$\vec{j}_1 = -en_0\vec{v}_1 = -en_e\vec{v}_1 \quad \Rightarrow \quad \frac{\partial \vec{j}_1}{\partial t} = -en_e\frac{\partial \vec{v}_1}{\partial t} = \frac{e^2 n_e}{m_e}\vec{E}_1$$

For a monochromatic wave of frequency ω with time dependence exp(-iωt)

$$\vec{v}_1 = -\frac{ie\vec{E}_1}{m_e\omega} \qquad \vec{j}_1 = \sigma\vec{E}_1 = \frac{ie^2 n_e}{m_e\omega}\vec{E}_1$$

Relative permittivity for the electromagnetic (transverse) wave ε_r^{tr}

$$\varepsilon_r^{tr} = 1 + \frac{i\sigma}{\varepsilon_0\omega} = 1 - \frac{e^2 n_e}{\varepsilon_0 m_e\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2} \qquad \text{(no spatial dispersion)}$$

When expression for current is inserted into wave equation

$$-\Delta \vec{E}_1 + \frac{1}{c^2}\frac{\partial^2 \vec{E}_1}{\partial t^2} = -\mu_0 \frac{\partial \vec{j}_1}{\partial t} = -\frac{1}{c^2}\frac{e^2 n_e}{\varepsilon_0 m_e}\vec{E}_1$$

The dispersion equation is obtained for a wave of the wavevector $\vec{k}$

$$\omega^2 = \omega_{pe}^2 + c^2 k^2$$

When ion current is added, then $\omega_{pe}^2 \rightarrow \omega_p^2$

$$k^2 = \omega^2 \mu_0 \varepsilon_0 \varepsilon_r^{\,tr} = \omega^2 (1 - \omega_p^2 / \omega^2) / c^2$$

phase $\mathrm{v}_\varphi = \dfrac{\omega}{k} = \sqrt{c^2 + \omega_p^2 / k^2}$ 　　　　 group $\mathrm{v}_g = \dfrac{\mathrm{d}\omega}{\mathrm{d}k} = c^2 k / \omega = c^2 / \mathrm{v}_\varphi$

for $\omega < \omega_p \quad \rightarrow \quad k^2 < 0$ 　　　 wave does not propagate, it penetrates into
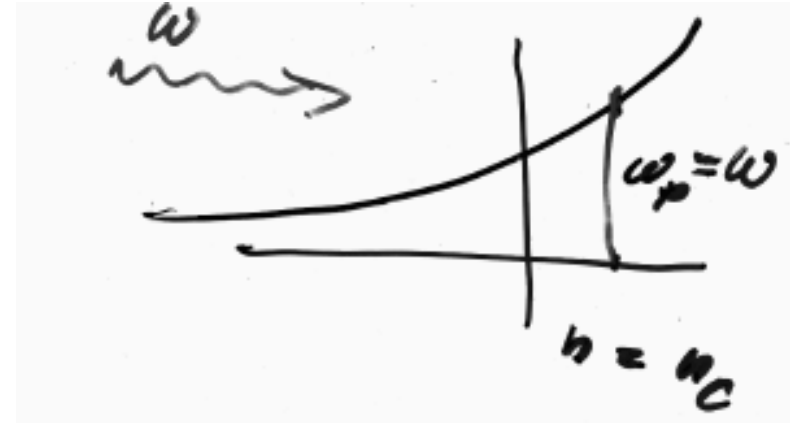plasma only via skin-effect

for $\omega \rightarrow \omega_{p+}$ 　　　 $k \rightarrow 0$ and wave is totally reflected (cut off frequency)

!! no absorption in collisionless approximation

$$n_c = \frac{\varepsilon_0 m_e \omega^2}{e^2}$$

$$\mathrm{Re}(\varepsilon_r) = 1 - \frac{n_e}{n_c}$$

n_c – critical density

$n_c = 10^{21} \mathrm{cm}^{-3}$ → λ=1,06 μm   (Nd-laser)

$n_c = 10^{19} \, \mathrm{cm}^{-3}$ → λ=10,6 μm   (CO_2-laser)

$n_c = 10^{13} \, \mathrm{cm}^{-3}$ → λ=1,06 cm   (cm waves)

Collisional absorption (also called inverse bremsstrahlung)

Electron collisions with other type of particles cause absorption (dominance of electron-ion collisions is assumed for simplicity)
Absorption of a photon during collision with ion is **time-reversed** process to the bremsstrahlung emission.

Derivation of absorption coefficient from microscopic description – tedious.
Macroscopically - collisions $\Rightarrow$ friction slowing down electron oscillations

When electron collides with ion, the ordered oscillation velocity of electron transforms to stochastic thermal velocity and wave must missing oscillation energy to the electron – and energy of wave is thus absorbed by electrons

Equation for velocity of electron fluid with electron-ion collision frequency ν_{ei}

$$\frac{\partial \vec{v}_1}{\partial t} + \nu_{ei} \vec{v}_1 = -\frac{e}{m_e} \vec{E}_1$$

For a monochromatic wave of frequency ω with time dependence exp(-iωt)

$$\vec{v}_1 = -\frac{ie\vec{E}_1}{m_e (\omega + i\nu_{ei})}$$

$$\vec{j}_1 = \sigma \vec{E}_1 = \frac{ie^2 n_e}{m_e (\omega + i\nu_{ei})} \vec{E}_1 = i\frac{\omega e^2 n_e}{m_e (\omega^2 + \nu_{ei}^2)} \vec{E}_1 + \frac{\nu_{ei} e^2 n_e}{m_e (\omega^2 + \nu_{ei}^2)} \vec{E}_1$$

Part of current in phase with the electric field $\Rightarrow$ Joule heating, absorption

Absorbed power per unit volume

$$W = \left\langle \vec{j}_1 \cdot \vec{E}_1 \right\rangle = \frac{1}{2} \Re \left\{ \vec{j}_1 \cdot \vec{E}_1^{\star} \right\} = \frac{1}{2} \frac{\nu_{ei} e^2 n_e}{m_e \left(\omega^2 + \nu_{ei}^2 \right)} |\vec{E}_1|^2$$

Relative permittivity - complex with positive imaginary part

$$\varepsilon_r^{tr} = 1 + \frac{i\sigma}{\varepsilon_0 \omega} = 1 - \frac{e^2 n_e}{\varepsilon_0 m_e \omega \left(\omega + i\nu_{ei} \right)} \simeq 1 - \frac{\omega_p^2}{\omega^2 + \nu_{ei}^2} + i \frac{\nu_{ei}}{\omega} \frac{\omega_p^2}{\omega^2 + \nu_{ei}^2}$$

Wavevector is then given by the expression

$$k^2 = \frac{\omega^2}{c^2} \varepsilon_r^{tr} = \frac{\omega^2}{c^2} \left(1 - \frac{\omega_p^2}{\omega^2 + \nu_{ei}^2} + i \frac{\nu_{ei}}{\omega} \frac{\omega_p^2}{\omega^2 + \nu_{ei}^2} \right)$$

Imaginary part-decrease of electric field amplitude in propagation direction

Absorbed energy is transferred to the **electron thermal energy**, ions are not directly heated. As energy exchange between electrons and ions is slow process, electron temperature in corona of laser-produced plasmas is higher than ion temperature.

Normal incidence of electromagnetic wave on planar plasma

Permittivity depends only on x ($\varepsilon_r = \varepsilon_r(x)$) and wavevector is in x direction

$$\text{curl } E + \frac{\partial B}{\partial t} = 0$$

$$\text{curl curl} = \text{grad div} - \Delta$$

$$\text{curl } B - \mu_0 \frac{\partial D}{\partial t} = 0$$

$$\text{div } \vec{D} = 0 = \varepsilon \text{ div } \vec{E} + \underbrace{\vec{E}\nabla \varepsilon}_{0} \qquad \Rightarrow \text{div } \vec{E} = 0$$

if time scale of density variations is $\tau >> \omega^{-1}$ $\qquad \varepsilon_r(x,t) \to \varepsilon_r(x)$ and

$$\Delta E - \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 E}{\partial t^2} = 0 \qquad \frac{\partial}{\partial t} \to -i\omega \qquad E \sim e^{-i\omega t}$$

$$\frac{\partial^2 E}{\partial x^2} + \frac{\omega^2}{c^2} \varepsilon_r E = 0 \quad \Rightarrow \quad k^2 = \frac{\omega^2}{c^2} \varepsilon_r \qquad \text{stationary wave equation}$$

$$\text{if } \lambda \left| \frac{\nabla \varepsilon}{\varepsilon} \right| << 1 \quad \Rightarrow \quad \varepsilon \text{ is slowly varying in space}$$

WKB approximation

$$E = E_+(x)e^{i\int k\,dx} + E_-(x)e^{-i\int k\,dx} \qquad E'' = \left[\underbrace{-k^2 E_+}_{0^{th}\,order} + \underbrace{2ik\frac{\partial E_+}{\partial x} + i\frac{\partial k}{\partial x}E_+}_{1^{st}\,order} + \underbrace{\frac{\partial^2 E_+}{\partial x^2}}_{2^{nd}\,order} \right]e^{i\int k\,dx} + ...$$

0th order $\qquad k^2 E_+ = \dfrac{\omega^2}{c^2}\varepsilon_r E_+ \qquad$ fulfilled

1.order $\qquad 2ik\dfrac{\partial E_+}{\partial x} + i\dfrac{\partial k}{\partial x}E_+ = 0 \qquad\qquad E \sim k^{-\frac{1}{2}} \sim \varepsilon_r^{-\frac{1}{4}}$

$$E = \frac{E_{0+}}{\sqrt[4]{\varepsilon_r}}e^{i\int k\,dx} + \frac{E_{0-}}{\sqrt[4]{\varepsilon_r}}e^{-i\int k\,dx} \qquad$$ WKB solution (no reflection !!)

Such density (permittivity) profiles do exist where WKB solves the task exactly

Critical point neighborhood – $\varepsilon \to 0$ - WKB approximation is not valid
Solution is found for linear profile of electron density n_e
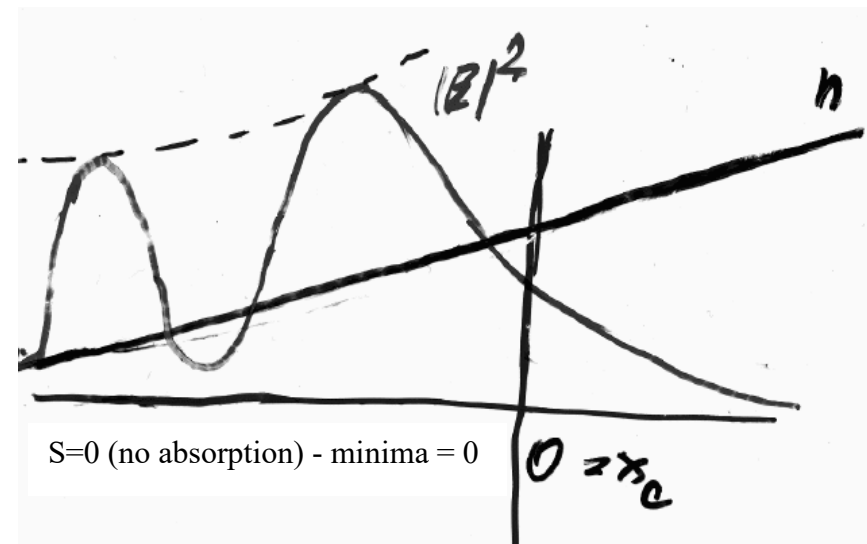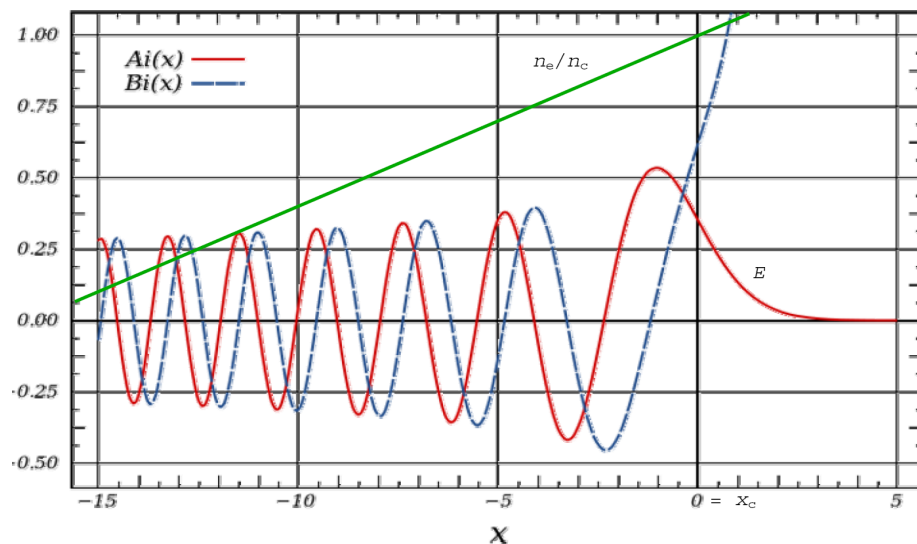relative permittivity $\varepsilon_r = -ax + iS$ where $S = \nu_{ei}/\omega$ in the critical surface

Plasma density is growing in direction of x axis – field must go to 0 for $x \to \infty$

$$\frac{\partial^2 E}{\partial x^2} + \frac{\omega^2}{c^2}\left(-ax + iS\right)E = 0 \qquad \xi = \left(\frac{\omega}{ca}\right)^{2/3}\left(-ax + iS\right) \implies \frac{d^2 E}{d\xi^2} + \xi E = 0$$

There exists exact solution that fulfills the boundary condition

$E = 3\,C\,\mathrm{Ai}\left(-\xi\right)$ \qquad Ai = <u>Airy function</u>

$$= \begin{cases} C\xi^{1/2}\left[\mathrm{J}_{1/3}\left(\frac{2}{3}\xi^{3/2}\right) + \mathrm{J}_{-1/3}\left(\frac{2}{3}\xi^{3/2}\right) \right] & \mathrm{Re}(\xi) > 0 \\[4ex] C\left(-\xi\right)^{1/2}\left[\mathrm{I}_{1/3}\left(\frac{2}{3}\left(-\xi\right)^{3/2}\right) + \mathrm{I}_{-1/3}\left(\frac{2}{3}\left(-\xi\right)^{3/2}\right) \right] & \mathrm{Re}(\xi) < 0 \end{cases}$$
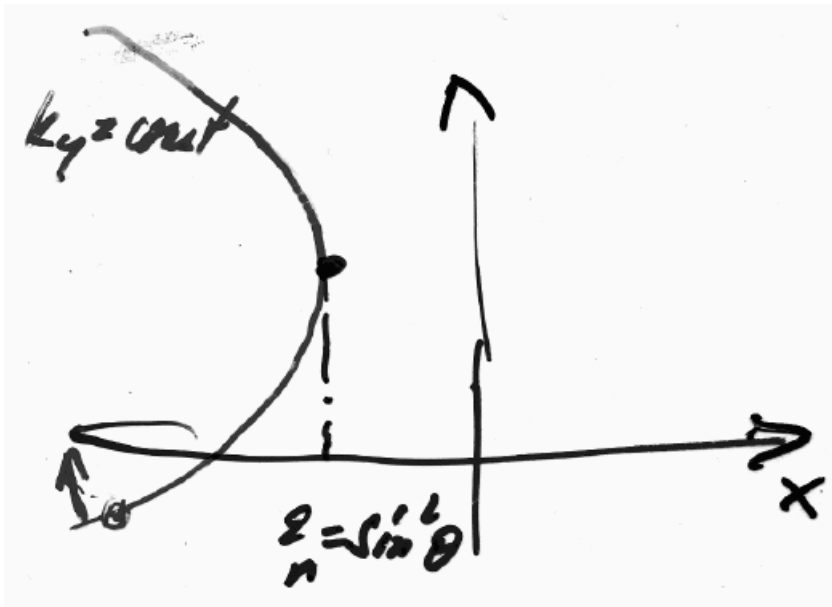

S=0 (no absorption) - minima = 0

Oblique incidence and resonance absorption

Permittivity depends only on x ($\varepsilon_r = \varepsilon_r(x)$) and wavevector $\vec{k} = \left(k_x, k_y, 0\right)$

$$k^2 = k_x^{\,2} + k_y^{\,2} = k_x^{\,2} + \frac{\omega^2}{c^2}\sin^2\theta_0$$

reflection point $\mathrm{Re}\left(\varepsilon_r\right) = \sin^2\theta_0$

$$
\left.\begin{array}{l}
\odot\, E \\[2mm]
\nwarrow\, B
\end{array}\right\} \quad \text{TE wave} =
$$

s-polarization

$$
\left.\begin{array}{l}
\odot\, B \\[2mm]
\nwarrow\, E
\end{array}\right\} \quad \text{TM wave} =
$$

p-polarization

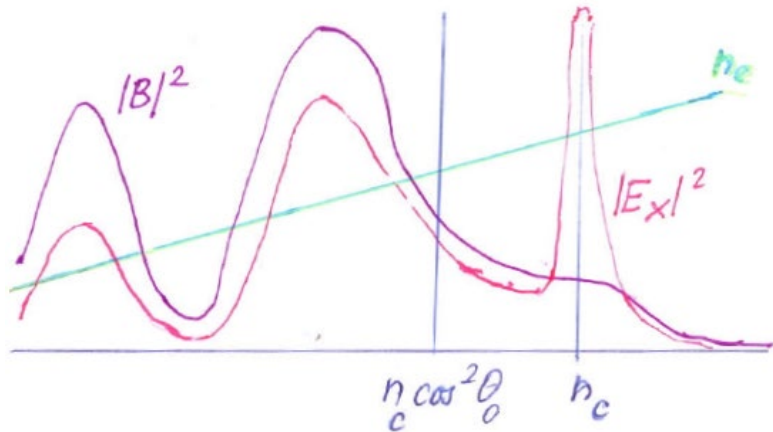p-polarization $\qquad \vec{E}\cdot\nabla\varepsilon \neq 0 \qquad$ div $\vec{E} \neq 0$

$$\frac{d^2 B}{dx^2} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dx}\frac{dB}{dx} + \frac{\omega^2}{c^2}\left(\varepsilon_r - \sin^2\theta_0\right)B = 0$$

$$E_x = -\frac{k_y B}{\omega\mu_0\varepsilon} = -\frac{\sin\theta_0}{\sqrt{\mu_0\varepsilon_0}}\cdot\frac{B}{\varepsilon_r} \qquad \text{singularity in the critical surface}$$

<u>Resonance absorption</u> $\quad t \to l$ (transverse elmg. wave transfers into longitudinal) - l – plasma wave cannot escape from plasma– collisional or collisionless absorption

in principle linear process – it exists even at small intensities I

when $\dfrac{\nu}{\omega} \to 0 \quad A = f(q) \qquad q = \left(k_0 L\right)^{2/3}\sin^2\theta_0$

$\theta_0 \to 0 \quad E_x\big|_{x_c} \to 0 \quad$ normal incidence – no E_x

$L \to \infty \quad E_x\big|_{x_c} \to 0 \quad$ reflection point far from x_c

$$\frac{B(x_c)}{B_0} \cong \frac{2(\pi \cos \theta_0)^{1/2}}{3^{1/3} \Gamma\left(1/3\right)(k_0 L)^{1/6}} \qquad \text{(for small } q\text{)}$$

$q \ll 1$

$$A = q\left(1 - \frac{2}{3}q\right)$$

$q \gg 1$

$$A = 2\exp\left(-\frac{4}{3}q^{3/2}\right) - \exp\left(-\frac{8}{3}q^{3/2}\right)$$

maximum $A \approx 0.5$ when $q \approx 0.65$
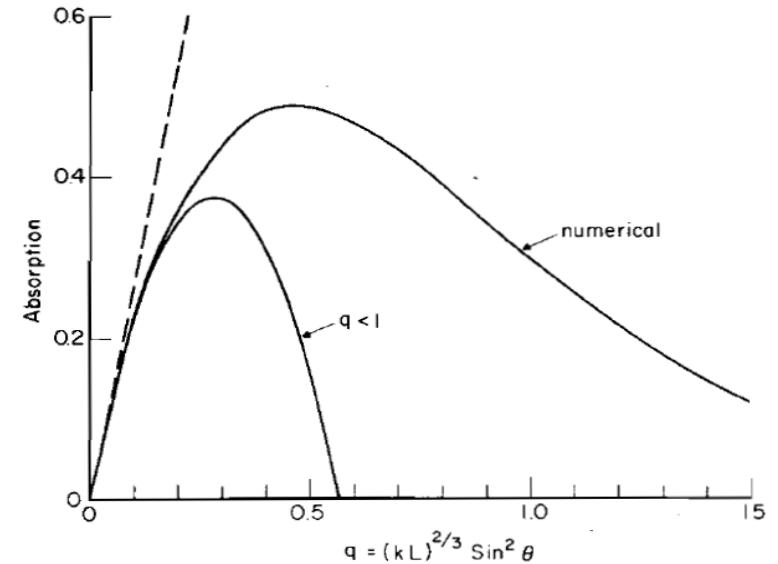
collisions $\varepsilon(x_c) = i\dfrac{v_c}{\omega}$ $E_z(x_c) = \dfrac{\sin \theta_0}{\dfrac{v_c}{\omega}} B(x_c) \cdot \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$

The width of field maximum Δ

$$\underbrace{\frac{|n-n_c|}{n_c}}_{\frac{\Delta}{L}} = \frac{\nu_c}{\omega} \quad \Rightarrow \quad \Delta = \frac{\nu_c}{\omega}L$$

Absorbed energy

$$W = \frac{\omega}{c}\int_{-\Delta/2}^{\Delta/2} \frac{\nu_{ei}}{\omega}|E|^2\,dz \simeq \frac{\nu_c}{\omega}\frac{\omega}{c}|E_z(x_c)|^2 \cdot \Delta = \frac{\omega}{c}\frac{\nu_c}{\omega}\frac{\sin^2\theta_0}{\left(\frac{\nu_c}{\omega}\right)^2}B^2(x_c)c^2 \cdot \left(\frac{\nu_c}{\omega}\right)L$$

$$W = \omega c B^2(x_c) \times L\sin^2\theta_0 \qquad \text{independent of } \nu_c$$

Warm plasma (spatial dispersion of longitudinal field)

$$\frac{1}{\varepsilon_0}\vec{D} = \varepsilon_r^{tr}\vec{E} + \frac{3v_{Te}^2}{c^2}\left[\text{grad div}\vec{E} - \frac{1}{3}\frac{\nabla\varepsilon_r}{\varepsilon_r - 1}\text{div}\vec{E}\right]$$

Plasma wave propagates from critical surface to rarified plasma, in decreasing density wavenumber k grows and thus v_φ decreases. When it becomes comparable to thermal velocity $\Rightarrow$ Landau damping (it accelerates electrons out of plasma)

plasma wave

growth k

Landau damping

absorption to fast electrons
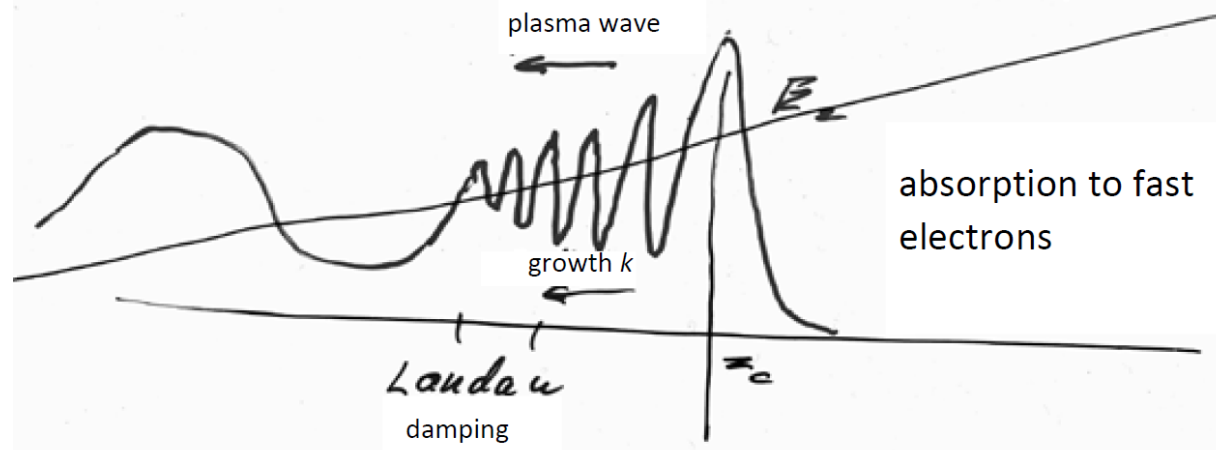
At higher intensities plasma wave damps via nonlinear mechanism of wavebreaking – energy is transferred to a small group of so called „hot (fast) electrons. Electrons are accelerated both into target, and to plasma-vacuum, where most of electrons are reflected in electrostatic field of sheath back into target.

Nonlinearities of electromagnetic wave propagation in plasmas

$$\varepsilon_r = 1 - \frac{e^2 n_e}{\varepsilon_0 m_e \omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

A. $\quad \to m_e$ — *relativistic nonlinearity*

$$m_e = \frac{m_{e0}}{\sqrt{1 - \dfrac{v^2}{c^2}}} \approx \frac{m_{e0}}{\sqrt{1 - \dfrac{e^2 |E_L|^2}{2 m_e^2 c^2 \omega^2}}} \qquad \text{(for } v_{osc} \gg v_{Te})$$

if $v_{osc} \ll c \Rightarrow$
$$\varepsilon_r = 1 - \frac{e^2 n_e}{\varepsilon_0 m_{e0} \omega^2} \left(1 - \frac{1}{4} \frac{e^2 |E_L|^2}{m_{e0}^2 c^2 \omega^2} \right)$$

nonlinearity $\quad \delta\varepsilon \sim |E_L|^2 / \omega^2 \sim I\lambda^2$ - quadratic nonlinearity – permittivity **grows** with the quadrat of electric field

B. $\rightarrow n_e$ - *density modification caused by ponderomotive force or pressure gradient*

a) ponderomotive nonlinearity

$$F_p = -\frac{\rho}{2\rho_c}\varepsilon_0 \nabla \langle E^2 \rangle = -\frac{\rho}{4\rho_c}\varepsilon_0 \nabla |E_L|^2$$

$$F_p - \nabla p = 0$$

Ponderomotive force pushes plasma out of intense field $\Rightarrow$ it causes density gradient $\Rightarrow$ pressure gradient

In equilibrium pressure gradient compensates for ponderomotive force:

$$-\frac{n_e}{4n_c}\varepsilon_0 \nabla |E_L|^2 - k_B T_e \nabla n_e = 0 \qquad \Rightarrow \qquad n_e = n_0 \exp\left(-\frac{\varepsilon_0 |E_L|^2}{4k_B T_e n_c}\right)$$

For small I – positive quadratic nonlinearity
$$\delta\varepsilon \sim |E_L|^2 / n_c \sim I\lambda^2$$

b) thermal – plasma is most heated in the field maximum and the density decreases there to keep the pressure constant

Nonlinearity consequences

A. Formation of filaments (hot spots inside a laser beam)

Quadratic medium $\varepsilon_R = \varepsilon_L + \varepsilon_2 \langle E^2 \rangle$

Wave $E = E_0 \cos(k_0 x - \omega t)$ $E \| z$ $\delta E \| z$

Perturbation (for simplicity periodic in the normal to E direction)

$$\delta E = \left[e_1(x) \cos(k_0 x - \omega t) + e_2(x) \sin(k_0 x - \omega t) \right] \cdot \cos(k_\perp y)$$

e_1 is the component in phase with the main planar wave

$$\varepsilon = \varepsilon_L + \varepsilon_2 \left\langle (E + \delta E)^2 \right\rangle = \underbrace{\varepsilon_L + \frac{1}{2} \varepsilon_2 E_0^2}_{\varepsilon_m} + \varepsilon_2 e_1(x) E_0 \cos(k_\perp y)$$

Field perturbation induces permittivity perturbation periodic in y

$$k_0{}^2 = \frac{\omega^2}{c^2}\varepsilon_m \qquad \text{wavevector in homogeneous medium}$$

notation

$$\Delta = \partial^2/\partial x^2 + (\partial^2/\partial y^2 + \partial^2/\partial z^2) = \nabla_\parallel^2 + \nabla_\perp^2$$

$$2ik_0\frac{\partial \delta E}{\partial x} + \nabla_\perp^2 \delta E + \frac{\omega^2}{c^2}\delta\varepsilon\, E_0 = 0 \qquad \text{wave equation for perturbation}$$

next rewritten for components

$$-2k_0\frac{de_2}{dx} - k_\perp{}^2 e_1 + \frac{\omega^2}{c^2}\varepsilon_2 E_0{}^2 e_1 = 0 \qquad \sim\cos(k_0 x - \omega t)$$

$$\sim\sin(k_0 x - \omega t)$$

$$2k_0\frac{de_1}{dx} - k_\perp{}^2 e_2 = 0$$

Assumption: $\qquad e_1(x) \sim e^{k_\parallel x} \qquad k_\parallel = \pm\frac{k_\perp}{2k_0}\sqrt{\frac{\omega^2}{c^2}\varepsilon_2 E_0{}^2 - k_\perp{}^2}$

The higher is the radiation intensity, the narrower perturbations can grow.

Maximum growth $k_{\parallel} = \dfrac{\omega \varepsilon_2 E_0{}^2}{4 c \sqrt{\varepsilon_m}}$ for $k_{\perp}{}^2 = \dfrac{\omega^2}{2c^2} \varepsilon_2 E_0{}^2$

Laser beam is finally split into high intensity zones (hot spots) – filaments are observed when looking from the side

B. *Self-focusing*

Laser beam focuses as a whole at lower intensity – filaments are formed earlier for high I and quadratic nonlinearity,

Wave equation (ε_L is linear and $\delta\varepsilon$ nonlinear part of permittivity)

$$\Delta E - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left[\left(\varepsilon_L + \delta\varepsilon \right) E \right] = 0$$

$$E = \Psi\left(x, y, z, \xi\right) \exp\left(ikx - i\omega t\right)$$

$$\xi = t - \frac{x}{v_g}$$

Ψ is slowly varying complex wave amplitude

$$\left(2ik\frac{\partial}{\partial x}+\nabla_{\perp}^{2}\right)\Psi=-k^{2}\frac{\delta\varepsilon}{\varepsilon_{L}}\Psi$$

parabolic equation

$k^2 = \omega^2 \varepsilon_L/c^2$

$$\Psi = A\exp(i\varphi)$$

Ψ is divided into real amplitude and phase

Then, one can write equations for the amplitude and the phase

$$k\frac{\partial}{\partial x}A^{2}=-\nabla_{\perp}\left(A^{2}\nabla_{\perp}\varphi\right)\;\ldots$$

describes the energy transport

$$\frac{\partial}{\partial x}\varphi+\frac{1}{2k}\left(\nabla_{\perp}\varphi\right)^{2}-\frac{k}{2}\left(\frac{\nabla_{\perp}^{2}A}{k^{2}A}+\frac{\delta\varepsilon}{\varepsilon}\right)=0$$

the shape of wavefront

diffraction   refraction

For collimated beam $\nabla_{\perp}\varphi=0$

Self-focusing of collimated beam occurs, if refraction is greater than diffraction

$$\frac{\delta\varepsilon}{\varepsilon} > -\frac{\nabla_{\perp}^{2}A}{k^{2}A}$$

For Gaussian beam

$$A = A_0 \exp\left(-\frac{r^2}{2a^2}\right)$$

In cylindric geometry

$$\nabla_{\perp}^{2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

and self-focusing condition reads

$$\frac{\varepsilon_2 A_0^{2}}{\varepsilon_L} > \frac{2}{k^2 a^2}$$

As $A_0^{2}a^2 \sim P$, self-focusing occurs, is beam power is $P > P_0$ – threshold power for self-focusing [for relativistic NL is $P_0 \simeq 17.5\left(n_c / n_e\right)$ GW].

Gaussian beam would focus according to the above equations ideally into a point focus, however the slowly varying amplitude approximation is not valid in its neighborhood.

Elmg. waves in plasma with external magnetic field B_0

Let's assume homogeneous stationary magnetic field B_0.
Plasma is **anisotropic**, we derive **conductivity tensor** σ_{ij} and from conductivity we shall calculate tensor of **high-frequency permittivity** ε_{ij}.
We shall assume high-frequency waves, and thus for simplicity we omit the ion current. Further, for simplicity we shall assume cold plasma and omit possible impact of gradient of high-frequency pressure.
Let axis z is in the direction of B_0. Let weak high frequency field E_1 (B_1 does not influence linear conductivity) is of frequency ω. Plasma is collisionless. Then

$$\vec{u}_1 = -i\,\frac{e}{m_e \omega}\,\vec{E}_1 - i\,\frac{\omega_c}{\omega}\,\vec{u}_1 \times \hat{z}$$

Magnetic field has no impact on conductivity in z direction and for directions x, y we obtain

$$u_{1x} = -i\,\frac{\omega e\, E_{1x}}{m_e\left(\omega^2 - \omega_c^2\right)} - \frac{e\,\omega_c E_{1y}}{m_e\left(\omega^2 - \omega_c^2\right)}$$

$$u_{1y} = \frac{e\,\omega_c E_{1x}}{m_e\omega^2\left(\omega^2 - \omega_c^2\right)} - i\,\frac{\omega e\, E_{1y}}{m_e\left(\omega^2 - \omega_c^2\right)}$$

The current is $\vec{j}_1 = -e\,n_{e0}\,\vec{u}_1 = \vec{\sigma}\,\vec{E}_1$ and permittivity $\varepsilon_{ij} = \varepsilon_0\,\varepsilon_{r,ij} = \varepsilon_0\left(\delta_{ij} + i\dfrac{\sigma_{ij}}{\omega\,\varepsilon_0}\right)$

$$\vec{\varepsilon}_r = \begin{pmatrix} 1 - \dfrac{\omega_p^2}{\omega^2 - \omega_c^2} & i\dfrac{\omega_c}{\omega}\dfrac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\[3mm] -i\dfrac{\omega_c}{\omega}\dfrac{\omega_p^2}{\omega^2 - \omega_c^2} & 1 - \dfrac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\[3mm] 0 & 0 & 1 - \dfrac{\omega_p^2}{\omega^2} \end{pmatrix}$$

From Maxwell's equations one derives wave equation

$$i\vec{k}\times\vec{E} - i\omega\vec{B} = 0 \qquad i\vec{k}\times\vec{B} + i\omega\varepsilon_0\vec{\varepsilon}_r\vec{E} = 0 \quad \Rightarrow \quad -\vec{k}\times\left(\vec{k}\times\vec{E}\right) - \dfrac{\omega^2}{c^2}\vec{\varepsilon}_r\vec{E} = 0$$

Coordinate system is chosen so that $\vec{k} = \left(k_x, 0, k_z\right)$ a non-zero solution of homogeneous equation exists only if the determinant is equal to 0.

$$\det \begin{pmatrix} k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{r,xx} & -\frac{\omega^2}{c^2}\varepsilon_{r,xy} & -k_x k_z \\ -\frac{\omega^2}{c^2}\varepsilon_{r,yx} & k_x^2 + k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{r,yy} & 0 \\ -k_x k_z & 0 & k_x^2 - \frac{\omega^2}{c^2}\varepsilon_{r,zz} \end{pmatrix} = 0$$

A. $k \perp B_0$ **(propagation normal to magnetic field)**

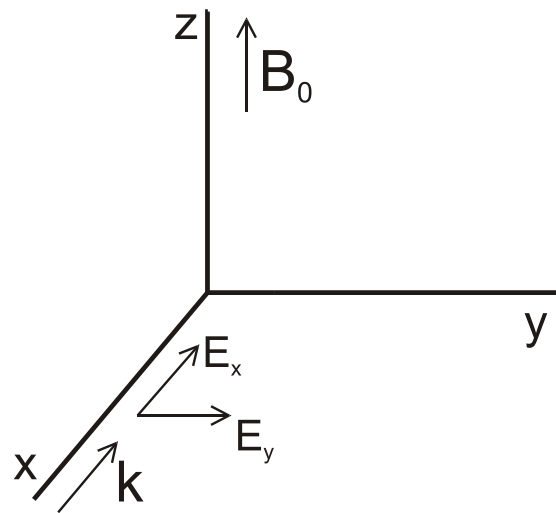if $\vec{E} \parallel \vec{B}_0$ B_0 does not influence wave – **ordinary wave-O**

<div align="right">

3rd line of determinant

$k_x^2 - \frac{\omega^2}{c^2}\varepsilon_{r,zz} = 0 \quad \Rightarrow \quad \omega^2 = \omega_p^2 + c^2 k^2$

</div>

extraordinary wave - X

influenced by field B_0– 1st and 2nd row and column

$\Rightarrow \; v_{1y} \Rightarrow F_x \sim v_{1y} \times B_0 \to v_{1x} \Rightarrow j_x \Rightarrow E_x$
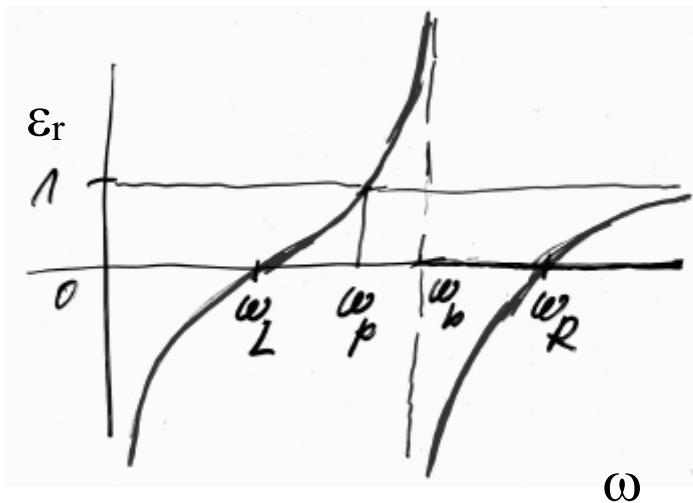
Electric field is not normal to the propagation direction

Dispersion relation for X wave is ($\omega_h^2 = \omega_p^2 + \omega_c^2$)

$$\frac{c^2 k^2}{\omega^2} = \frac{c^2 k_x^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}$$

Relative permittivity of plasma for X wave (indexu of refraction n squared)

$$\varepsilon_r = n^2 = 1 - \frac{\omega_p{}^2}{\omega^2} \frac{\omega^2 - \omega_p{}^2}{\omega^2 - \omega_h{}^2}$$

permittivity $\varepsilon_r = \pm\infty$ pro $\omega = \omega_h$

permittivity $\varepsilon_r = 0$ for $\omega_L = \dfrac{1}{2}\left[-\omega_c + \sqrt{\omega_c{}^2 + 4\omega_p{}^2} \right]$

and for $\omega_R = \dfrac{1}{2}\left[\omega_c + \sqrt{\omega_c{}^2 + 4\omega_p{}^2} \right]$

for $\omega \to \omega_{h-}$ there is resonance ($k \to \infty$) $\Rightarrow$ total absorption of wave

points $\omega \to \omega_{L+}, \omega_{R-}$ are cutoffs ($k \to 0$) $\Rightarrow$ total reflection of the wave

B. $k \parallel B_0$ **(propagation along magnetic field)**

$$\det \begin{pmatrix} k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{r,xx} & -\frac{\omega^2}{c^2}\varepsilon_{r,xy} \\ -\frac{\omega^2}{c^2}\varepsilon_{r,yx} & k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{r,yy} \end{pmatrix} = 0$$
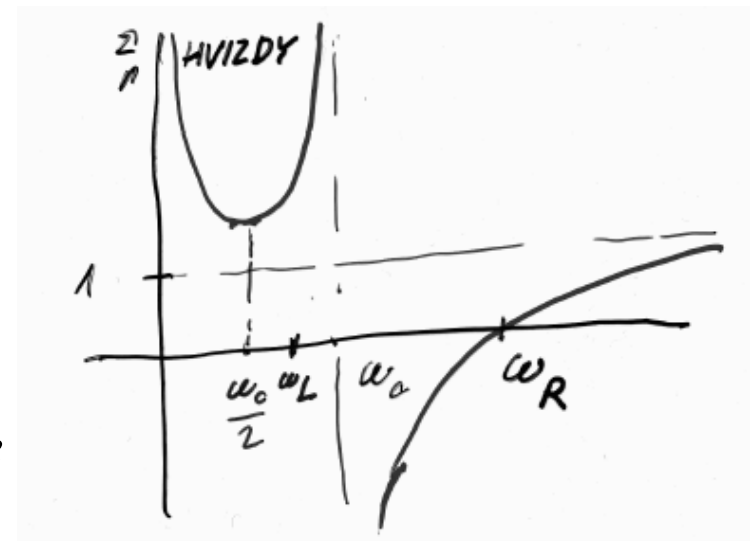
Left-handed circular wave (L wave)

$$\varepsilon_r = n^2 = 1 - \frac{\omega_p^2}{\omega^2}\frac{1}{1 + \frac{\omega_c}{\omega}}$$ ω_L - cutoff, propagation for $\omega > \omega_L$

Right-handed circular wave (R wave) – cyclotron rotation of electrons is right-handed - resonance!!

$$\varepsilon_r = n^2 = 1 - \frac{\omega_p^2}{\omega^2}\frac{1}{1 - \frac{\omega_c}{\omega}}$$ propagates for $\omega > \omega_R$

and $\omega < \omega_c$ (whistlers - hvizdy)– resonance for ω_c

permittivity for R wave

CMA diagram

(electromagnetic waves)

It contains

- boundaries between particular areas of propagation types
- vertically – propagation along B, horizontally – normally to B
- dependence of phase velocity on direction of propagation
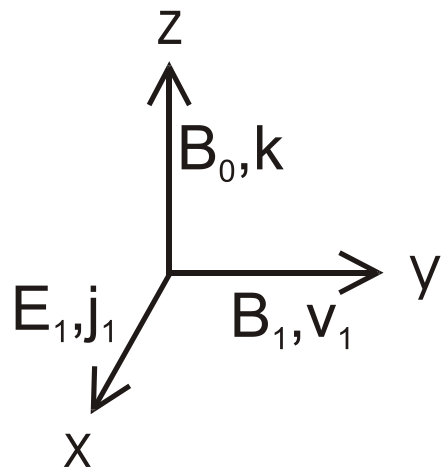- transition between types of waves if propagation direction changes

density (ω_p^2/ω^2) on horizontal axis
magnetic field oriented along vertical
axis – value (ω_c/ω) on vertical axis
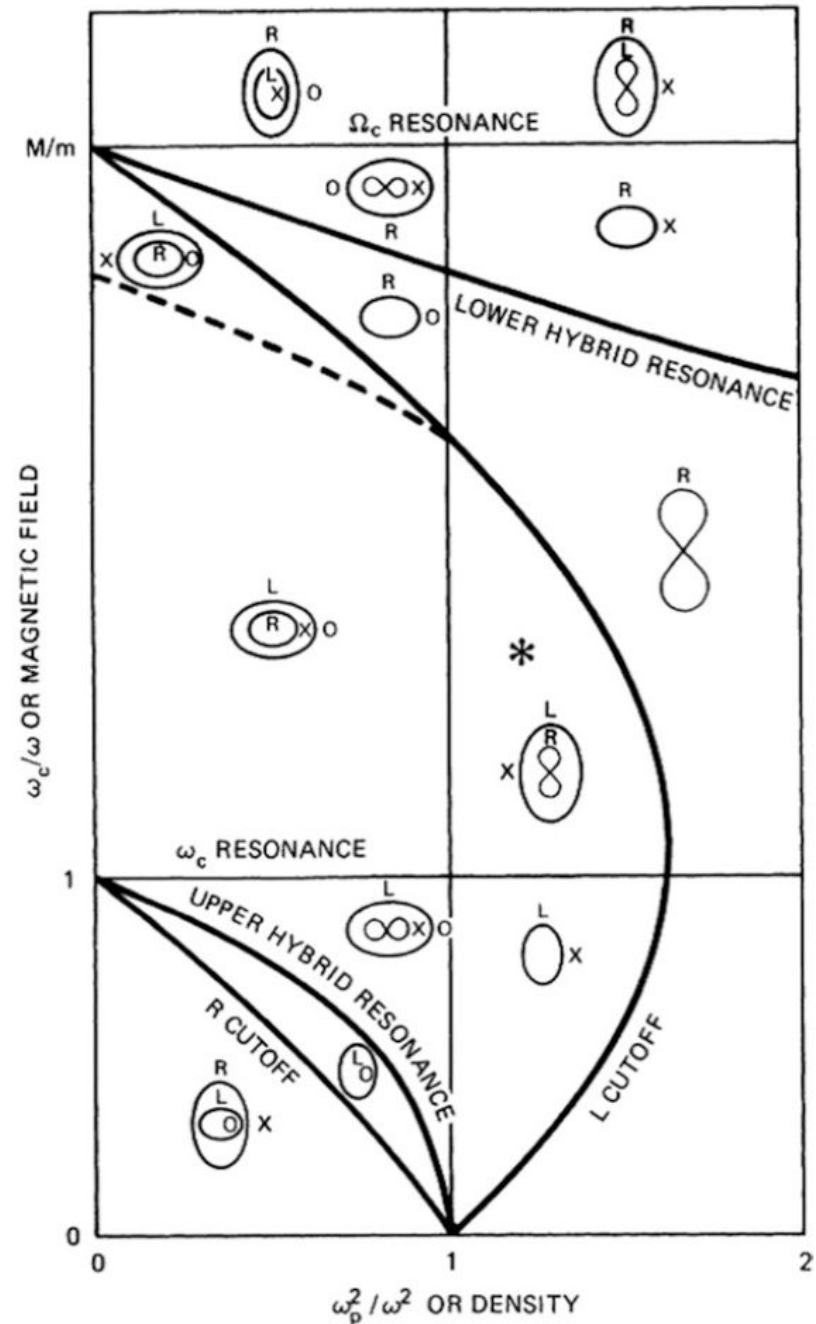
Ion electromagnetic waves (hydromagnetic waves)

– exist only in presence of B_0

a) $\vec{k} \parallel \vec{B}_0$ – **Alfvén wave**

z

B_0, k

E_1, j_1 B_1, v_1 y

x

$i\vec{k} \times \vec{E}_1 = i\omega\vec{B}_1$

electron density

$$\frac{1}{\mu_0} i\vec{k} \times \vec{B}_1 + i\omega\varepsilon_0\vec{E}_1 = n_0 e\left(\vec{v}_i - \vec{v}_e\right)$$

$$0 = -e\vec{E}_1 - e\vec{v}_e \times \vec{B}_0$$

$$-i\omega M_i\vec{v}_i = Ze\vec{E}_1 + Ze\vec{v}_i \times \vec{B}_0 \qquad \text{equation of motion}$$

$$v_{ex} = 0 \qquad v_{ey} = \frac{\vec{E}_1 \times \vec{B}_0}{B_0^{\,2}} \qquad \text{E×B drift}$$

$$v_{iy} = -i\frac{ZeB_0}{\omega M_i} v_{ix} = -i\frac{\Omega_c}{\omega} v_{ix} \qquad \Omega_c = \frac{ZeB_0}{M_i} \qquad \text{ion cyclotron frequency}$$

$$v_{ix} = i \frac{ZeE_{1x}}{\omega M_i \left(1 - \frac{\Omega_c^2}{\omega^2}\right)} \qquad \omega_{pi}^2 = \frac{Ze^2 n_0}{\varepsilon_0 M_i} \qquad \text{ion plasma frequency}$$

We substitute current into wave eq. $\Rightarrow$ dispersion rel. $\quad 1 - \dfrac{k^2 c^2}{\omega^2} + \dfrac{\omega_{pi}^2}{\Omega_c^2 - \omega^2} = 0$

for $\omega^2 \ll \Omega_c^2$ $\qquad \omega^2 = \dfrac{k^2 v_A^2}{1 + \left(\dfrac{v_A}{c}\right)^2}$ where $\quad v_A^2 = \dfrac{c^2 \Omega_c^2}{\omega_{pi}^2} = \dfrac{c^2 B_0^2 \varepsilon_0}{\rho_M}$

Alfvén velocity

$$Y_B(z,t) = \int^z \frac{B_y(z',t)}{B_0} dz' \qquad v_B = -i\omega Y_B = -\frac{\omega}{k}\frac{B_y}{B_0} \qquad B_y = \frac{k}{\omega} E_x$$
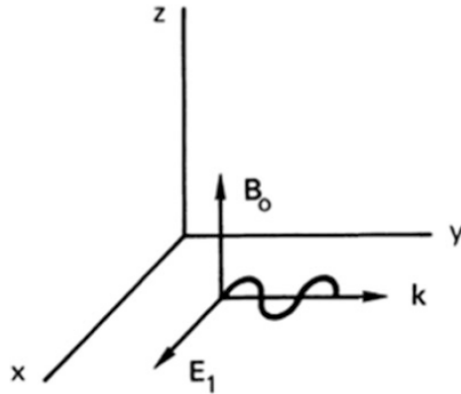
Y_B shift of field line position, v_B velocity of field line motion

$$\Rightarrow \quad v_B = -\frac{E_x}{B_0} = v_D$$

Plasma moves with the same velocity as the field line
Magnetic field line are „frozen" in plasma"

Alfvén wave – transverse waves „on string" String $\cong$ magnetic field line
(Hannes Alfvén – Nobel prize – 1970)

b) $\vec{k} \perp \vec{B}_0$ – magnetosonic wave $\vec{E} \perp \vec{B}_0$

dispersion relation for cold plasma

$$\omega^2 = \frac{k^2 v_A{}^2}{1 + \frac{v_A{}^2}{c^2}} \approx k^2 v_A{}^2$$

for $T_e \neq 0$ additional action of pressure $\qquad \omega^2 = k^2 \frac{v_A{}^2 + c_s{}^2}{1 + \frac{v_A{}^2}{c^2}} \approx k^2 \left(v_A{}^2 + c_s{}^2 \right)$