Plasma as a mixture of fluids

(suggested reading – D.R. Nicholson, Introduction to plasma theory, §7.1, 7.2)

Fluid equations (hydrodynamic two-fluid equations)

particles of type „s“ (s = e^-, i^+), collisionless plasma

$$\frac{\partial f_s}{\partial t} + \vec{v}\frac{\partial f_s}{\partial \vec{r}} + \frac{q_s}{m_s}\left(\vec{E} + \vec{v}\times\vec{B}\right)\frac{\partial f_s}{\partial \vec{v}} = 0$$

Vlasov equation

$$n_s = \int f_s \mathrm{d}\vec{v}$$

$$\vec{v}_s = \frac{1}{n_s}\int f_s \vec{v}\,\mathrm{d}\vec{v}$$

density n_s and average velocity $\vec{v}_s$

0th moment — integral $\int \mathrm{d}\vec{v}$ of Vlasov equation

$$\frac{\partial n_s}{\partial t} + \nabla\left(n_s \vec{v}_s\right) = 0$$

continuity equation (conservation of particle number)

1st moment integral $m_s \int \vec{v}\, d\vec{v}$ of Vlasov equation

$$\vec{V} = \vec{v} - \vec{v}_s \qquad \rho_s = m_s n_s$$

$$\frac{\partial}{\partial t}\left(m_s n_s v_{si}\right) + \frac{\partial}{\partial r_j}\left(m_s n_s v_{si} v_{sj}\right) + \frac{\partial}{\partial r_j}\underbrace{\left(m_s n_s \left\langle V_i V_j \right\rangle\right)}_{P_{ij}^s} = n_s F_{si}$$

$$n_s F_{si} \rightarrow q_s\left(\vec{E} + \vec{v}_s \times \vec{B}\right)$$

conservation of momentum law $\quad P_{ij}^s \rightarrow$ pressure tensor

Pressure is tensor $P_{ij}^s = p^s \delta_{ij} + \Pi_{ij}^s$, where Π_{ij} is viscous pressure – $\mathrm{tr}(\Pi_{ij}) = 0$

$$\frac{\partial \vec{v}_s}{\partial t} + \left(\vec{v}_s \nabla\right)\vec{v}_s = -\frac{1}{\rho_s}\,\mathrm{div}\,\overleftrightarrow{P}^s + \frac{\vec{F}_s}{m_s}$$

force equation (Navier-Stokes eq.)

Including collisions into equation of motion

Mutual collisions of particles of the same sort – no impact on $\vec{\mathrm{v}}_s$

for $t \neq s$ $\quad -\nu_{st}\left(\vec{\mathrm{v}}_s - \vec{\mathrm{v}}_t\right)$… braking by friction against particles t

$$\frac{\partial \vec{\mathrm{v}}_s}{\partial t} + \left(\vec{\mathrm{v}}_s \nabla\right)\vec{\mathrm{v}}_s = -\frac{1}{\rho_s}\,\mathrm{div}\,\overleftrightarrow{P}^s + \frac{\vec{F}_s}{m_s} - \sum_t \nu_{st}\left(\vec{\mathrm{v}}_s - \vec{\mathrm{v}}_t\right)$$

it follows from momentum conservation law that

$$m_s \nu_{st}\left(\vec{\mathrm{v}}_s - \vec{\mathrm{v}}_t\right) + m_t \nu_{ts}\left(\vec{\mathrm{v}}_t - \vec{\mathrm{v}}_s\right) = 0$$

$$\Rightarrow \nu_{ts} = \frac{m_s}{m_t}\nu_{st} = \frac{m_s}{m_s + m_t}\nu_{st}^*$$

Energy conservation law

simplified assumption are often used

- adiabatic process $p = Cn^{\gamma}$
- isothermal process $\gamma = 1$

to avoid solving equation for temperature (heat conduction)

Derivation of energy conservation via 2nd moment of Vlasov equation

$$\int \frac{1}{2} m_s \mathrm{v}^2 d\vec{\mathrm{v}} \qquad \frac{1}{2} m_s \int \mathrm{V}^2 f_s d\vec{\mathrm{V}} = \frac{3}{2} n_s k_B T_s \qquad (\vec{\mathrm{V}} = \vec{\mathrm{v}} - \vec{\mathrm{v}}_s)$$

Heat flux $$\vec{q}_s = \frac{1}{2} m_s \int \vec{\mathrm{V}} \mathrm{V}^2 f_s d\vec{\mathrm{V}}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}n_s k_B T_s\right)+\mathrm{div}\left\{\vec{q}_s+\vec{\mathrm{v}}_s\,\frac{3}{2}n_s k_B T_s\right\}+P_{ik}^s\,\frac{\partial \mathrm{v}_{si}}{\partial r_k}=0$$

$$\frac{3}{2}n_s k_B\,\frac{\partial T_s}{\partial t}+\frac{3}{2}n_s k_B\left(\vec{\mathrm{v}}_s\nabla\right)T_s+\mathrm{div}\vec{q}_s+\underbrace{P_{ik}^s\,\frac{\partial \mathrm{v}_{si}}{\partial r_k}}=0$$

work by pressure

if $P_{ik}^s=\delta_{ik}\,p^s\Rightarrow p^s\,\mathrm{div}\,\vec{\mathrm{v}}_s$ work by scalar pressure

One-fluid approximation

uses mass density ρ, average mass velocity **v**, temperatures may differ T_e, T_i

in magnetic field – magnetohydrodynamics *(will be described later)*

for the description of laser-produced plasmas, quasineutrality approximation is applied and one obtains one-fluid two-temperature hydrodynamics

Drift motions of fluid

$\vec{v} \perp B$ for any particles with m, q

$$mn\left[\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\nabla\right)\vec{v}\right] = qn\left(\vec{E} + \vec{v} \times \vec{B}\right) - \nabla p$$

∇p was absent in 1 particle description

slow motions => $\omega << \omega_c$

$\frac{\partial \vec{v}}{\partial t}$ is omitted, because $\left|\frac{mn\frac{\partial \vec{v}}{\partial t}}{qn\,\vec{v} \times \vec{B}}\right| \approx \left|\frac{mn\omega v_\perp}{qn v_\perp B}\right| = \frac{\omega}{\omega_c} << 1$

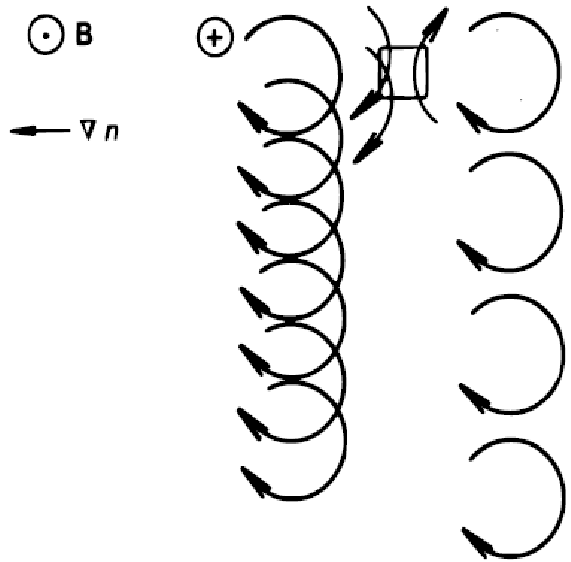
assumption $\left(\vec{v}\nabla\right)\vec{v} \simeq 0$ - term including velocity square – small for small speeds

$$0 = qn\left(\vec{E} + \vec{v} \times \vec{B}\right) - \nabla p \qquad \Big| \times \vec{B}$$

$$\left(\vec{v} \times \vec{B}\right) \times \vec{B} = \vec{B}\left(\vec{v} \cdot \vec{B}\right) - \vec{v}B^2 = -\vec{v}_\perp B^2$$

=> $$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}$$ $\vec{E} \times \vec{B}$ drift

$$\vec{v}_D = -\frac{\nabla p \times \vec{B}}{qnB^2}$$ …diamagnetic drift



explanation of diamagnetic drift

$\vec{v}_D \perp \nabla p$ - then $\left(\vec{v}\nabla\right)\vec{v}$ is often exactly = 0

$$\vec{j}_D = n_e e\left(\vec{v}_{Di} - \vec{v}_{De}\right) = \frac{B \times \nabla\left(p_i + p_e\right)}{B^2}$$ diamagnetic current

$$n_i = \frac{n_e}{Z} \qquad q_i = Ze \qquad q_e = -e \qquad p_e \approx n_e k_B T_e \qquad p_i \approx n_i k_B T_i = \frac{n_e}{Z} k_B T_i$$

curvature drift - same as in guiding center approx., but grad B drift is absent !!!
inhomogeneous E – drift is different from that in guiding center approximation

$\vec{\mathrm{v}} \parallel \vec{B}$

$\vec{B} = \left(0, 0, B_z\right) \to \mathrm{v}_{\parallel} = \mathrm{v}_z$

$\mathrm{v}_z \frac{\partial}{\partial z} \mathrm{v}_z$ – is often omitted (slow motion and small gradient)

$$\frac{\partial \mathrm{v}_z}{\partial t} = \frac{q}{m} E_z - \frac{\gamma k_B T}{mn} \frac{\partial n}{\partial z}$$

If the right side were large for e$^-$ $\to \frac{\partial \mathrm{v}_z}{\partial t}$ also large

$\frac{\partial n_e}{\partial z} \neq \frac{\partial (Z n_i)}{\partial z}$ → quasineutrality $n_e \simeq Z n_i$ violation

=> $$-eE_z = \frac{\gamma_e k_B T_e}{n_e} \frac{\partial n_e}{\partial z} \equiv e \frac{\partial \Phi}{\partial z}$$

slow motion $\gamma_e = 1$

=> $e\Phi = k_B T_e \ln n_e + C$ => $n_e = n_0 \exp\left(e\Phi \big/ k_B T_e \right)$

In plasmas – quasineutrality principle

$n_e = Zn_i$ $\wedge$ $\vec{E} \neq 0$

$\vec{E} = -\frac{k_B T_e}{en_e} \frac{\partial n_e}{\partial z}$ $\vec{E}$ is calculated from ∇n_e, and not from Poisson equation

This is called **plasma approximation**